Classical wave equation scalar normal modes and Green functions for confocal hyperboloidal electrodes

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The scalar normal modes and Green functions for the classical wave equation subject to Dirichlet and Neumann boundary conditions on confocal hyperboloidal surfaces, which model the shapes of the electrodes in a scanning tunneling microscope or a conductor-insulator-conductor junction, are explicitly constructed. These modes and functions are of interest as possible starting points for the study of the Casimir effect between the electrodes of such devices.

Keywords: Classical wave equation; normal modes; Green functions; scanning tunneling microscopy; Casimir effect.

Se construyen explícitamente los modos normales y las funciones de Green escalares para la ecuación de onda clásica sujetas a condiciones de frontera de Dirichlet y de Neumann en superficies hiperboloidales confocales, las cuales modelan las formas de los electrodos en un microscopio de tunelaje y barrido, y en juntas conductor-aislante-conductor. Estos modos y funciones son de interés como posibles puntos de partida para el estudio del efecto Casimir entre los electrodos de tales dispositivos.

Descriptores: Ecuación de onda clásica; modos normales; funciones de Green; microscopía de barrido y tunelamiento; efecto Casimir.

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1. Introduction

The modeling of the tip and sample surfaces in scanning tunneling microscopy (STM) as confocal hyperboloids has been used in the recent literature [1-3]. Such a modeling can also be extended for conductor-insulator-conductor (CIC) junctions [4,5]. The quantitative description of the electrostatic potential, the electric intensity field and the electrostatic energy distribution between the electrodes in both devices, taking into account the applied bias voltage and the charges induced in the electrodes while the tunneling electron is in transit between the latter, involves the solutions of the Laplace equation and the Green function for the Poisson equation subject to Dirichlet boundary conditions [2,4,5].

Casimir analyzed the quantum electromagnetic vacuum energy change in the presence and absence of one of two uncharged parallel plane conductors, and concluded that there should be a net attractive force between the latter [6]. Experimental measurements have confirmed the existence of this so-called Casimir effect [7]. The effect has also been investigated for other geometries of the electrodes [8]. Some of the possible starting points for the analysis of the Casimir effect for electrodes of a given shape require the scalar normal modes or the Green functions for the classical wave equation subject to Dirichlet and Neumann boundary conditions at the surface of the electrodes [9].

In this work, motivated by the interest in the study of the Casimir effect for hyperboloidal electrodes, the corresponding normal modes and Green functions are constructed using prolate spheroidal coordinates. The task is accomplished in two successive steps. In Sec. 2, the solutions of the homogeneous classical wave equation in those coordinates are identified in general, and selected according to Dirichlet and Neumann boundary conditions in particular. The orthonormal bases of these Dirichlet and Neumann solutions are used in Sec. 3 to construct the respective Green functions, as the particular solutions of the inhomogeneous wave equation with a point and instantaneous source. Section 4 presents a discussion of the analogies and differences between the problem and results of this paper and those of the electrostatic situation [5], and also the remaining and anticipated steps for the quantum analysis of the Casimir effect for the STM and CIC-junction geometries. In the Appendix some basic properties of the angular and radial spheroidal functions needed in Secs. 2 and 3 are included.

2. Solutions of the homogeneous wave equation in spheroidal coordinates

The prolate spheroidal coordinates $(1 \leq \eta < \infty, -1 \leq \xi \leq 1, 0 \leq \varphi \leq 2\pi)$ are defined by the transformation equations to Cartesian coordinates,

$$
\begin{aligned}
x &= f\sqrt{(\eta^2-1)(1-\xi^2)}\cos\varphi, \\
y &= f\sqrt{(\eta^2-1)(1-\xi^2)}\sin\varphi, \\
z &= f\eta\xi,
\end{aligned}
\tag{1}
$$

where $2f$ is the distance between the focii located at $(x=0, y=0, z=\pm f)$. Constant values of the respective coordinates correspond to confocal prolate spheroids with eccentricity $1/\eta$, confocal hyperboloids with eccentricity $1/\xi$, and meridian half-planes making an angle φ with the xz plane[10, 11] . The respective scale factors

$$
\begin{aligned}
h_\eta &= f\sqrt{\frac{\eta^2-\xi^2}{\eta^2-1}}, \\
h_\xi &= f\sqrt{\frac{\eta^2-\xi^2}{1-\xi^2}}, \\
h_\varphi &= f\sqrt{(\eta^2-1)(1-\xi^2)},
\end{aligned}
\tag{2}
$$

and the orthogonal unit vectors

$$
\begin{aligned}
\hat{\eta} &= \frac{(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi)\eta\sqrt{1-\xi^2} + \hat{k}\xi\sqrt{\eta^2-1}}{\sqrt{\eta^2-\xi^2}}, \\
\hat{\xi} &= \frac{-(\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi)\xi\sqrt{1-\eta^2} + \hat{k}\eta\sqrt{1-\xi^2}}{\sqrt{\eta^2-\xi^2}}, \\
\hat{\varphi} &= -\hat{\imath}\sin\varphi + \hat{\jmath}\cos\varphi,
\end{aligned}
\tag{3}
$$

follow from the evaluation of the differential displacement vector. Then, the homogeneous wave equation in these coordinates has the form

$$
\left\{ \frac{1}{f^2(\eta^2-\xi^2)}\left[\frac{\partial}{\partial\eta}(\eta^2-1)\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\right] \right.
$$
$$
\left. + \frac{1}{f^2(\eta^2-1)(1-\xi^2)}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\psi(\eta,\xi,\varphi,t)=0. \tag{4}
$$

It clearly admits separable solutions

$$
\psi(\eta,\xi,\varphi,t) = H(\eta)\Xi(\xi)\Phi(\varphi)T(t) \tag{5}
$$

in which each factor satisfies the respective differential equations:

$$
\left[\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta} - \frac{m^2}{\eta^2-1} + \frac{\omega^2 f^2}{c^2}\eta^2\right]H(\eta)=\lambda H(\eta), \tag{6}
$$

$$
\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi} - \frac{m^2}{1-\xi^2} - \frac{\omega^2 f^2}{c^2}\xi^2\right]\Xi(\xi)=-\lambda\Xi(\xi), \tag{7}
$$

$$
\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi, \tag{8}
$$

$$
\frac{d^2T}{dt^2} = -\omega^2 T, \tag{9}
$$

where ω^2, m^2 and λ are the successive separation constants. The time dependent solution is chosen to vary harmonically with frequency ω, $T = e^{-i\omega t}$. The periodicity condition on the azimuth angle dependent function, $\Phi(\varphi + 2\pi) = \Phi(\varphi)$, determines the integer values of the separation constant, $m = 0, \pm 1, \pm 2, \ldots$, as well as the form of the function itself, $e^{im\varphi}$ or its alternatives $\cos(m\varphi)$ and $\sin(m\varphi)$. The combination $\omega/c = k$ is identified as the wave number.

Equations (6) and (7) have the same form, but their solutions are defined in the respective domains, $1 \leq \eta < \infty$ and $-1 \leq \xi \leq 1$, and correspond to the radial and angular spheroidal functions [11, 12]. When both domains are entirely available, the respective spheroidal functions can be written as series of Bessel functions and associated Legendre functions of integer orders, involving the same expansion coefficients and the same characteristic values of the separation constant λ. The latter can be calculated via the orthodox continued fraction equation method [11], or by an equivalent matrix method [12].

The normal mode solutions of the homogeneous wave equation, Eq. (4), to be selected are the one subjected to the Dirichlet boundary conditions of vanishing at the surfaces of the two hyperboloidal electrodes considered in this work,

$$
\psi_D(\eta, \xi=\xi_1, \varphi, t) = 0, \quad \psi_D(\eta, \xi=\xi_2, \varphi, t) = 0, \tag{10}
$$

and to the Neumann boundary conditions of vanishing normal derivatives on the same surfaces,

$$
\left.\frac{\partial\psi_N(\eta,\xi,\varphi,t)}{\partial\xi}\right|_{\xi=\xi_1} = 0, \quad \left.\frac{\partial\psi_N(\eta,\xi,\varphi,t)}{\partial\xi}\right|_{\xi=\xi_2} = 0. \tag{11}
$$

The separable solutions of Eq.(5) and the boundary conditions of Eqs. (10) and (11) lead to the restrictions on the respective solutions of Eq. (7):

$$
\Xi_D(\xi=\xi_1) = 0, \quad \Xi_D(\xi=\xi_2) = 0, \tag{12}
$$

and

$$
\Xi'_N(\xi=\xi_1) = 0, \quad \Xi'_N(\xi=\xi_2) = 0, \tag{13}
$$

where the prime denotes the derivative with respect to ξ.

In order to construct these functions, let us consider the general solutions of Eq. (7), for chosen values of m, kf and λ. They are simply the superpositions of the angular spheroidal functions of the first and second kinds [11],

$$
\Xi^b_{mn}(\xi) = A^b_{mn}S^{(1)}_{mn}(kf,\xi) + B^b_{mn}S^{(2)}_{mn}(kf,\xi), \tag{14}
$$

given by Eq. (A.1) and (A.2). For $b = D$, Eqs. (12) become

$$
A^D_{mn}S^{(1)}_{mn}(kf,\xi_1) + B^D_{mn}S^{(2)}_{mn}(kf,\xi_1) = 0,
$$
$$
A^D_{mn}S^{(1)}_{mn}(kf,\xi_2) + B^D_{mn}S^{(2)}_{mn}(kf,\xi_2) = 0, \tag{15}
$$

a set of two linear homogeneous algebraic equations for the unknown coefficients A_{mn}^D y B_{mn}^D. The latter may be different from zero only if the determinant of Eqs.(15) vanishes

$$S_{mn}^{(1)}(kf, \xi_1) S_{mn}^{(2)}(kf, \xi_2)$$
$$-S_{mn}^{(2)}(kf, \xi_1) S_{mn}^{(1)}(kf, \xi_2) = 0, \quad (16)$$

which will happen only for an infinite set of discrete wave numbers k_{mns}^D, $s = 1, 2, 3, \ldots$, with corresponding frequen-

cies $\omega_{mns}^D = k_{mns}^D c$, and eigenvalues $\lambda_{mn}^D(k_{mns}^D f)$. For each eigenfrequency, Eqs. (15) and (16) determine the ratios of the coefficients in the solution of Eq.(14):

$$\frac{A_{mn}^D}{B_{mn}^D} = -\frac{S_{mn}^{(2)}(k_{mns}^D f, \xi_1)}{S_{mn}^{(1)}(k_{mns}^D f, \xi_1)} = -\frac{S_{mn}^{(2)}(k_{mns}^D f, \xi_2)}{S_{mn}^{(1)}(k_{mns}^D f, \xi_2)}. \quad (17)$$

Then, the Dirichlet angular spheroidal functions can be written in two alternative forms

$$\Xi_{mns}^D(kf, \xi) = N_{mns}^D \left[S_{mn}^{(2)}(k_{mns}^D f, \xi_2) S_{mn}^{(1)}(k_{mns}^D f, \xi) - S_{mn}^{(1)}(k_{mns}^D f, \xi_2) S_{mn}^{(2)}(k_{mns}^D f, \xi) \right]$$
$$= \bar{N}_{mns}^D \left[S_{mn}^{(1)}(k_{mns}^D f, \xi_1) S_{mn}^{(2)}(k_{mns}^D f, \xi) - S_{mn}^{(2)}(k_{mns}^D f, \xi_1) S_{mn}^{(1)}(k_{mns}^D f, \xi) \right] \quad (18)$$

exhibiting that they are solutions of Eq.(7) and satisfy both boundary conditions of Eqs.(12). The eigenvalue nature of the problem defined by these equations guarantees that the basis of the functions of Eq.(18) is orthogonal and complete. The normalization constants can be chosen so that

$$\int_{\xi_1}^{\xi_2} d\xi \Xi_{nms}^D(k_{mns}^D f, \xi) \Xi_{n'ms'}^D(k_{mns'}^D f, \xi) = \delta_{nn'}\delta_{ss'}, \quad (19)$$

$$\sum_n \sum_s \Xi_{nms}^D(k_{mns}^D f, \xi) \Xi_{nms}^D(k_{mns}^D f, \xi') = \delta(\xi - \xi'). \quad (20)$$

Similarly, for $b = N$, Eqs.(13) take the forms

$$A_{mn}^N S_{mn}^{(1)'}(kf, \xi_1) + B_{mn}^N S_{mn}^{(2)'}(kf, \xi_1) = 0,$$
$$A_{mn}^N S_{mn}^{(1)'}(kf, \xi_2) + B_{mn}^N S_{mn}^{(2)'}(kf, \xi_2) = 0. \quad (21)$$

The coefficients A_{mn}^N and B_{mn}^N have nontrivial and nonvanishing values only for

$$S_{mn}^{(1)'}(kf, \xi_1) S_{mn}^{(2)'}(kf, \xi_2)$$
$$-S_{mn}^{(2)'}(kf, \xi_1) S_{mn}^{(1)'}(kf, \xi_2) = 0, \quad (22)$$

corresponding to discrete sets of values for the wave number k_{mns}^N, $s = 1, 2, 3, \ldots$, frequencies $\omega_{mns}^N = k_{mns}^N c$, and eigenvalues $\lambda_{nm}^N(k_{mns}^N f)$.

Now the ratio of the coefficients in the solution of Eq.(14) are

$$\frac{A_{mn}^N}{B_{mn}^N} = -\frac{S_{mn}^{(2)'}(k_{mns}^N f, \xi_1)}{S_{mn}^{(1)'}(k_{mns}^N f, \xi_1)} = -\frac{S_{mn}^{(2)'}(k_{mns}^N f, \xi_2)}{S_{mn}^{(1)'}(k_{mns}^N f, \xi_2)}, \quad (23)$$

and the Neumann orthonormal angular spheroidal functions take the alternative forms

$$\Xi_{mns}^N(\xi) = N_{mns}^N \left[S_{mn}^{(2)'}(k_{mns}^N f, \xi_2) S_{mn}^{(1)}(k_{mns}^N f, \xi) - S_{mn}^{(1)'}(k_{mns}^N f, \xi_2) S_{mn}^{(2)}(k_{mns}^N f, \xi) \right]$$
$$= \bar{N}_{mns}^N \left[S_{mn}^{(1)'}(k_{mns}^N f, \xi_1) S_{mn}^{(2)}(k_{mns}^N f, \xi) - S_{mn}^{(2)'}(k_s^N f, \xi_1) S_{mn}^{(2)}(k_{mns}^N f, \xi) \right]. \quad (24)$$

The counterparts of Eqs.(19) and (20) expressing the orthonormality and the completeness of the Neumann basis are obtained by the replacement $D \rightarrow N$.

The radial spheroidal functions $R_{mn}^{(p)}(kf, \eta)$ of Eq. (6), to be used in Eq. (5), are given by Eqs.(A.5-8), as superpositions of spherical Bessel functions of kind $p = 1, 2, 3, 4$, involving the same coefficients and eigenvalues appearing in the angular spheroidal functions.

In conclusion, two infinite discrete sets of separable solutions of the wave equation, of the form of Eq.(5), have been identified in this section:

$$\psi^D(\eta, \xi, \varphi, t) = R_{mn}^{(1)}(k_{mns}^D f, \eta) \Xi_{mns}^D(\xi)$$
$$\times e^{im\varphi} e^{-iw_{mns}^D t}, \quad (25)$$

$$\psi^N(\eta, \xi, \varphi, t) = R_{mn}^{(1)}(k_{mns}^N f, \eta) \Xi_{mns}^N(\xi)$$
$$\times e^{im\varphi} e^{-iw_{mns}^N t}, \quad (26)$$

for $b = D$ and N boundary conditions. Here the radial mode with $p = 1$, involving the regular spherical Bessel functions, is chosen to represent stationary spheroidal waves between

the hyperboloidal electrodes. Other choices are possible depending on the situation at hand.

3. Construction of the Green functions

The Green functions are solutions of the inhomogeneous wave equation with a unit, point and instantaneous source,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_i(\vec{r},t;\vec{r}',t') = -4\pi\delta(\vec{r}-\vec{r}')\delta(t-t'), \quad (27)$$

and are subjected to the respective Dirichlet and Neumann boundary conditions

$$G_D(\vec{r},t;\vec{r}',t')|_{\vec{r}'\varepsilon S} = 0, \quad (28)$$

$$\frac{\partial G_N}{\partial n'}(\vec{r},t;\vec{r}',t')|_{\vec{r}'\varepsilon S} = 0, \quad (29)$$

on the surface of the electrodes. Next, the construction of these Green functions is implemented for the hyperboloidal geometry.

Since the right hand side of Eq.(27) is zero at all points of space and instants of time, except at the point $\vec{r} = \vec{r}'$ and instant $t = t'$ in which the source is present, the Green functions can be constructed using the known solutions of the homogeneous Eq.(4). The space dependent Dirac-delta function in spheroidal coordinates has the form

$$\delta(\vec{r}-\vec{r}') = \frac{\delta(\eta-\eta')\delta(\xi-\xi')\delta(\varphi-\varphi')}{h_\eta h_\xi h_\varphi}. \quad (30)$$

The complete orthonormal sets of eigenfunctions in the hyperboloidal, azimuthal and time independent variables, solutions of Eqs.(7), (8) and (9), studied in the previous section, allow the representations

$$\delta(\xi-\xi')\delta(\varphi-\varphi')\delta(t-t') = \sum_m\sum_n\sum_s \Xi^b_{mns}(\xi)$$

$$\times\Xi^b_{mns}(\xi')\frac{e^{im(\varphi-\varphi')}}{2\pi}\frac{e^{-i\omega^b_{mns}(t-t')}}{2\pi}, \quad (31)$$

for $b = D$ and N.

The same functions are used in the expansions of the Green functions

$$G_b(\eta,\xi,\varphi,t;\eta',\xi',\varphi',t') = \sum_m\sum_n\sum_s g^b_{mns}(\eta,\eta')$$

$$\times\Xi^b_{mns}(\xi)\Xi^b_{mns}(\xi')\frac{e^{im(\varphi-\varphi')}}{2\pi}\frac{e^{-i\omega^b_{mns}(t-t')}}{2\pi} \quad (32)$$

ensuring that the boundary conditions of Eqs. (28) and (29) are satisfied because the hyperboloidal coordinate dependent functions satisfy Eqs.(12) and (13), respectively.

The substitution of Eqs.(30),(31) and (32) in Eq.(27) using the differential wave operator of Eq.(4), leads to the ordinary differential equation for the η dependent expansion "coefficients" of Eq.(32):

$$\left[\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta} - \frac{m^2}{\eta^2-1} + k^{b\,2}_{mns}f^2\eta^2 - \lambda^b_{mn}(k^b_{mns}f)\right]$$

$$\times g^b_{mns}(\eta,\eta') = -4\pi\delta(\eta-\eta') \quad (33)$$

which is the inhomogeneous version of the radial Eq.(6). Its solutions are constructed from the appropriate combinations of radial spheroidal functions $R^{(p)}_{mn}(k^b_{mns}f,\eta)$, Eqs.(A.5-8). It is necessary to distinguish between $\eta < \eta'$ and $\eta > \eta'$, and to use the symmetry of the Green functions under the exchange of the field and source positions and times, and their continuity at $\vec{r} = \vec{r}'$, $t = t'$. These conditions lead to the choices

$$g^{b(-)}_{mns}(\eta\le\eta') = C^b_{mns}R^{(3)}_{mn}(k^b_{mns}f,\eta')R^{(1)}_{mn}(k^b_{mns}f,\eta) \quad (34)$$

$$g^{b(+)}_{mns}(\eta\ge\eta') = C^b_{mns}R^{(1)}_{mn}(k^b_{mns}f,\eta')R^{(3)}_{mn}(k^b_{mns}f,\eta) \quad (35)$$

involving the regular ($p = 1$) and the outgoing ($p = 3$) spheroidal radial functions Eqs. (A.5,6,7).

Integration of Eq.(33) around $\eta = \eta'$ allows us to recognize the discontinuity of the derivative of its solution at that position,

$$(\eta^2-1)\frac{dg^b_{mns}(\eta,\eta')}{d\eta}\bigg|_{\eta=\eta'_+} - (\eta^2-1)\frac{dg^b_{mns}(\eta,\eta')}{d\eta}\bigg|_{\eta=\eta'_+} = -4\pi \quad (36)$$

Substitution of Eqs.(34) and (35) in Eq.(36) leads to the determination of the coefficients C^b_{mns}:

$$C^b_{mns}(\eta^2-1)\left[R^{(1)}_{mn}(k^b_{mns}f,\eta')R^{(3)'}_{mn}(k^b_{mns}f,\eta')\right.$$

$$\left. -R^{(3)}_{mn}(k^b_{mns}f,\eta')R^{(1)'}_{mn}(k^b_{mns}f,\eta')\right] = -4\pi \quad (37)$$

The quantity inside the brackets is identified as the Wronskian of the radial functions of the first and third kinds. Its value follows from Eqs. (A.11,12) and leads to the coefficients

$$C^b_{mms} = 4\pi i/k^b_{mns}f. \quad (38)$$

The final form of the Green functions of Eq. (32) follows from Eqs. (34), (35) and (38):

$$G_b(\eta,\xi,\varphi,t;\eta',\xi',\varphi',t') = \frac{4\pi i}{f}\sum_m\sum_n\sum_s\frac{1}{k^b_{mns}}je_{mn}(k^b_{mns}f,\eta_<)he^{(1)}_{mn}(k^b_{mns}f,\eta_>)$$

$$\Xi^b_{mn}(k^b_{mns}f,\xi)\Xi^b_{mn}(k^b_{mns}f,\xi')\frac{e^{im(\varphi-\varphi'}}{2\pi}\frac{e^{-i\omega^b_{mns}(t-t')}}{2\pi}, \quad (39)$$

using the Morse and Feschbach convention for the radial spheroidal wave functions, and the hyperboloidal wave functions of Eqs. (18) and (22) for $b = D$ and N.

Their time Fourier transforms can be evaluated in a straightforward way:

$$G_b(\eta,\xi,\varphi;\eta',\xi',\varphi';\omega) = \int\limits_{-\infty}^{\infty} G_b(\eta,\xi,\varphi,t;\eta',\xi',\varphi',t,)e^{i\omega(t-t')}d(t-t') = \frac{4\pi i}{fc}$$

$$\times \sum_m \sum_n \sum_s je_{mn}(k_{mns}^b f,\eta_<)he_{mn}^{(1)}(k_{mns}^b f,\eta_>)\Xi_{mn}^b(k_{mns}^b f,\xi)\Xi_{mn}^b(k_{mns}^b f,\xi')\frac{e^{im(\varphi-\varphi')}}{2\pi}\frac{\delta(\omega-\omega_{mns}^b)}{\omega_{mns}^b}, \quad (40)$$

exhibiting the discretization of the frequency spectra.

4. Discussion

The scalar normal modes and Green functions for the classical wave equation between confocal hyperboloidal electrodes have been constructed in Secs. 2 and 3. The discussion of this section is focussed on three specific points: the general results of this work and their concrete applications to the cases of two symmetric hyperboloidal and one plane and one hyperboloidal electrodes, which are geometries of special interest in CIC-junction and STM devices. It is also instructive to compare the electrostatic and electromagnetic situations for the latter, recognizing the analogies and differences in their equations and solutions. Finally, the results of Secs. 2 and 3 which serve as input data for the analysis of the Casimir effect, in the respective mode summation and Green function methods, are identified.

The scalar normal modes constructed in Sec. 2 for Dirichlet and Neumann boundary conditions are represented by Eqs. (25) and (26), respectively. The explicit forms of the hyperboloidal wavefunctions are given by Eqs. (18) and (24), and the corresponding characteristic frequencies and eigenvalues are evaluated via Eqs. (16) and (22).

In the specific case of symmetric hyperboloidal electrodes, for which $\xi_2 = -\xi_1$, the hyperboloidal wavefunctions have a definite parity and reduce to either $S_{mn}^{(1)}(kf,\xi)$ or $S_{mn}^{(2)}(kf,\xi)$ in Eq.(14), with the consequent particularizations in the subsequent equations. Similarly, in the case of a plane electrode $\xi_2 = 0$ and a hyperboloidal one $0 < \xi_1 < 1$, the hyperboloidal wavefunctions in Eq.(14) reduce to either $S_{mn}^{(1)}(kf,\xi)$ or $S_{mn}^{(2)}(kf,\xi)$, and additionally only odd (even) values of r in Eqs. (A.1) or (A.2) need to be included for Dirichlet (Neumann) boundary conditions, respectively. In both cases Eqs.(16) and (22) are simplified.

The Dirichlet and Neumann Green functions were explicitly constructed in Sec. 3 using the respective normal mode bases, with the result of Eq. (39). Their spectral representation of Eq. (40) was also obtained.

Both this work and Ref. 5 deal with hypeboloidal electrodes, using prolate spheroidal coordinates to describe their shapes and fields. The electromagnetic and electrostatic situations, studied in each case, are based on the homogeneous and inhomogeneous wave equations, and the Laplace and Poisson equations, respectively. The normal modes and Green functions in this work are constructed using spheroidal functions, Eqs.(25) and (26), while the harmonic functions in Ref. 5 are simply and directly associated Legendre functions of the first and second kinds. Both situations involve simpler solutions for the CIC-junction and STM geometries.

Casimir evaluated the change of the quantum electromagnetic vacuum energy,

$$\frac{1}{2}\left(\sum \hbar\omega\right)_I - \frac{1}{2}\left(\sum \hbar\omega\right)_{II}, \quad (41)$$

for plane electrodes at I) a finite separation and II) an infinite separation, where the first term includes the discrete summation over the characteristic frequencies for the parallel neighboring electrodes, and the second term becomes and integral over the continuum of frequencies associated with II. He introduced a cut-off function and used the Euler-Mclaurin formula to implement the summation over both terms [6]. For the hyperboloidal electrodes, the characteristic frequencies ω_{mns}^b for both the Dirichlet, Eq. (16), and Neumann, Eq. (22), modes, have to be included in the I-term.

The sum of the zero-point energies of the modes can also be rewritten as a space and frequency integral of the spectral Green function, Eq. (40),

$$\frac{1}{2}\sum_a \hbar\omega_a^b e^{-i\omega_a\tau}\bigg|_{\tau\to 0} = \frac{\hbar}{i}\int d^3x$$

$$\times \int \frac{d\omega}{2\pi}\omega^2 G_b(\vec{r},\vec{r}';\omega)e^{-i\omega(t-t')}\bigg|_{\vec{r}\to\vec{r}',t\to t'} \quad (42)$$

using the regularization techniques [9]. For additional QED techniques and their applications to the analysis of the Casimir effect, the reader is referred to Refs. 8 and 9.

A Appendix

The properties and representations of the angular and radial spheroidal functions, presented here for completeness sake, can also be found in [11-14]. The angular spheroidal functions of the first and second kinds are the linearly independent solutions of Eq.(7), expressed as linear combinations of associated Legendre polynomials P_{m+r}^m, and functions Q_{m+r}^m of the second kind, respectively, with common coefficients and eigenvalues evaluated by matrix methods described in the

same references:

$$S_{mn}^{(1)}(kf, \xi) = \sum_{r=0 \, or \, 1}^{\infty} d_r^{mn}(kf) P_{m+r}^m(\xi), \qquad (A.1)$$

$$S_{mn}^{(2)}(kf, \xi) = \sum_{r=-\infty}^{\infty} d_r^{mn}(kf) Q_{m+r}^m(\xi), \qquad (A.2)$$

and are associated with the successive eigenvalues of the separation constant, $\lambda_{mn}(kf)$.

The radial spheroidal functions $R_{mn}^{(p)}(kf, \eta)$ of the p-kind, $p = 1, 2$, are proportional to those of Eqs.(A.1) and (A.2), respectively, with the replacement of the angular variable ξ by the radial variable η. The joining of the angular and radial functions is given by

$$S_{mn}^{(1)}(kf, \eta) = \kappa_{mn}^{(1)} R_{mn}^{(1)}(kf, \eta), \qquad (A.3)$$

$$S_{mn}^{(2)}(kf, \eta) = \kappa_{mn}^{(2)} R_{mn}^{(2)}(kf, \eta), \qquad (A.4)$$

including the explicit forms of the joining factors $\kappa_{mn}^{(p)}$ in the Flammer convention [11,13].

While the convergence of Eqs. (A.1) and (A.2) is assured for $-1 \leq \xi \leq 1$, the same does not hold for $\eta > 1$. A better representation is a series of spherical Bessel functions:

$$R_{mn}^{(p)}(kf, \eta) = \left\{ \sum_{r=0,1}^{\infty} \frac{(2m+r)!}{r!} d_r^{mn} \right\}^{-1} \left(\frac{\eta^2 - 1}{\eta^2} \right)^{\frac{m}{2}}$$

$$\sum_{r=0,1}^{\infty} i^{r+m-n} \frac{(2m+r)!}{r!} d_r^{mn}(kf) z_{m+r}^{(p)}(kf\eta), \quad (A.5)$$

where

$$z_l^{(1)}(x) = j_l(x), \qquad z_l^{(2)}(x) = y_l(x) \qquad (A.6)$$

There are also radial spheroidal functions of the $p = 3$ and 4 kinds involving the corresponding outgoing and incoming spherical Hankel functions:

$$z_l^{(3)}(x) = h_l^{(1)}(x) = j_l(x) + i\, n_l(x), \qquad (A.7)$$

$$z_l^{(4)}(x) = h_l^{(2)}(x) = j_l(x) - i\, n_l(x). \qquad (A.8)$$

Equation (6) can be written explicitly for the radial spheroidal wave functions of the first and second kinds, in order to evaluate its Wronskian:

$$\left[\frac{d}{d\eta}(\eta^2 - 1)\frac{d}{d\eta} - \frac{m^2}{\eta^2 - 1} - k^2 f^2 \eta^2 - \lambda \right]$$
$$\times R_{mn}^{(1)}(kf, \eta) = 0, \quad (A.9)$$

$$\left[\frac{d}{d\eta}(\eta^2 - 1)\frac{d}{d\eta} - \frac{m^2}{\eta^2 - 1} - k^2 f^2 \eta^2 - \lambda \right]$$
$$\times R_{mn}^{(2)}(kf, \eta) = 0. \quad (A.10)$$

By multiplying Eq.(A.9) by $R_{mn}^{(2)}(kf, \eta)$ and Eq.(A.10) by $R_{mn}^{(1)}(kf, \eta)$, substracting them and integrating in the vicinity of η, the result is

$$(\eta^2 - 1)\left[R_{mn}^{(2)}(kf, \eta) \frac{dR_{mn}^{(1)}(kf, \eta)}{d\eta} \right.$$
$$\left. - R_{mn}^{(1)}(kf, \eta) \frac{dR_{mn}^{(2)}(kf, \eta)}{d\eta} \right] = \text{constant} \quad (A.11)$$

The net result is that the Wronskian of the radial spheroidal wavefunctions is inversely proportional to $(\eta^2 - 1)$. The value of the proportionality constant depends on the normalization convention for the radial functions themselves.

In the Morse and Feschbach convention and notation [14]:

$$je_{mn}(kf, \eta)\frac{dne_{mn}(kf, \eta)}{d\eta}$$
$$- \frac{dje_{mn}(kf, \eta)}{d\eta}ne_{mn}(kf, \eta) = \frac{1}{kf(\eta^2 - 1)} \quad (A.12)$$

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