Teaching logic: Cracking the hard nut

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Abstract
Two questions are addressed in this article: 1) How to make the students realize the importance of logic; and 2) how to teach the logical rules. The teacher may begin their logic class with an attempt to answer 1. Logic studies and records the basic moves of intelligence. When it analyses an argument *A*, it splits *A* into small steps. If each unit step seems to be intuitively right, then we accept *A* to be a valid argument. This "splitting" is the special skill of the logician. This skill helps one evaluate an ordinary argument in our day-to-day life. Question 2 is directly related to the didactics of logic. One may teach the rules of logic by demonstrating fallacies, i.e., by comparing the rules with their corresponding non-rules. If the teacher shows how the violation of a rule leads one to an intuitively undesired conclusion the student, learns the importance of rules.

Keywords
Didactics, logic, *logica utens*, *logica docens*, fallacies, logic in High School.

Enseñar lógica: Romper la nuez dura

Resumen
Este artículo aborda dos preguntas: 1) ¿Cómo lograr que los estudiantes se den cuenta de la importancia de la lógica?; y 2) ¿Cómo enseñar las reglas lógicas? El profesor podría comenzar la clase de lógica intentando responder la pregunta 1. La lógica estudia y registra los movimientos básicos de la inteligencia. Si desde esta disciplina se analiza un argumento *"A"*, se procede a dividirlo en pasos pequeños. Si cada uno de esos fragmentos unitarios parecen ser intuitivamente correctos, entonces aceptamos que *A* es un argumento válido. Esta "divisi*ón*" es la habilidad especial del conocedor de la lógica y nos ayuda a evaluar argumentos ordinarios en nuestra vida cotidiana. La pregunta 2 está directamente relacionada con la didáctica de la lógica. Uno podría enseñar las reglas de la lógica demostrando falacias, por ejemplo, comparando reglas con sus correspondientes antireglas. Si el profesor demuestra cómo la violación de una regla nos lleva a una conclusión que intuitivamente no es deseable, entonces el alumno aprende la importancia de las reglas.

Palabras clave
Didáctica, lógica, *logica utens*, *logica docens*, falacias, enseñanza de la lógica en el bachillerato.

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Teaching logic: Cracking the hard nut

Teaching logic is doubly difficult. First of all, it is as rigorous as any other analytical study, e.g., mathematics. Secondly, it is not easy to convince your students that logic is important. Every student knows that they have to take their mathematical lessons seriously. But many of us have to face the following question quite frequently in our logic class: "Why should we learn logic?" I think a logic course should begin with an attempt to answer this question. And very often teachers do face another problem in a logic class: when they start teaching the basic rules, such as *Modus Ponens* (if *p* then *q*; *p*; therefore *q*) some of their students would definitely ask "Are those rules to be learned? We know all of that anyway". In the beginning the logic lessons seem to be deceptively naïve. Then you start Predicate Calculus. Students will have a great amount of difficulties understanding the restrictions imposed on the rules. Some will fail to understand them and hence soon lose interest in learning logic any more. In this article, I shall basically share my teaching experience with those of my colleagues who teach logic. I will try to address the following distinct but mutually related questions: 1) Why should one take logic seriously?; 2) How should one handle the problems that are faced while teaching senior secondary students symbolic logic?

Why logic?

P. T. Geach (1979) tells us that medieval writers used to make a distinction between *logica utens*, "the practice of thinking logically about this or that subject-matter", and *logica docens*, "the construction of logical theory" (p. 5). There are two ways of learning: mechanical and cordial. Mechanical learning is algorithmic; "you do this, then you do that, and then. . . . and you solve the problem". No understanding is involved in this process. Cordial learning means "understanding something". In the developing world, most of the education systems encourage the mechanical process. It is easy to handle, as well as effective – temporarily, of course. On the other hand, cordial learning demands more time and involvement; it allows time for the *click* of understanding to emerge. Any analytical science is based on arguments. Mechanical learning does not try to see the argumentative threads in a scientific discourse; it helps solving problems. What helps us understand something or learn something cordially is *logica utens*.

It is not difficult to discover that "*logica utens* needs the aid of *logica docens*" (Geach, 1979, p. 6). Why? Logic is based on both intuition and technique. It is a map of our intelligence. The basic laws such as *Modus Ponens* correspond to the basic moves of intelligence. They are "intuitively" right for (almost) everybody.

When one sees a long argument one may not have an intuitive judgment about its (in)correctness. If one is logically trained then one knows how to split it into smaller pieces. If each piece is intuitively right, i.e., if each piece is a licensed logical move, then the argument is valid. Had logic been completely intuitive, there would have been no space for proving a theorem with heuristic devices. In that case, one should be able to respond to an argument intuitively just by looking at it. The logician (who is trained in *logica docens*) knows how to cut down a long argument into smaller, intelligible pieces. Analysis is nothing but this "logical splitting". It is not completely mechanical, since the minimum logical moves, i.e., the basic rules are always intuitive. One has to make decisions while solving a problem, because there are infinite options.

Epp (2003) and Bakó (2002) observe that many students of pure mathematics cannot write proofs properly; it is mainly because they fail to see the logical moves that underlie the steps of a mathematical proof. Following my previous statements, logic pictures the moves of intelligence and creates a flowchart that helps a learner understand how the $(n + 1)^{th}$ line of a proof (or argument) follows from the n^{th} line. In that sense, logic simulates intelligence.

Our *logica docens* is still not able to analyze all intuitive procedures. Still it does not mechanically trace the steps of *all* the arguments that seem to be perfectly valid. But *logica docens* is enriched every day, like all other disciplines. The main point is that logic aims to understand and record even the most minuscule intuitive moves.

Teaching logic at high schools

I have found that a few techniques help high-school students understand symbolic logic in a non-mechanical way. This discussion will be confined to propositional and predicate calculi. I'm sure these techniques can be further extended.

Learning rules

Natural science has an advantage over logic; the former is empirical to a great extent. You have something in flesh and bone to show to your younger students. But the laboratory of logic is our own mind. How can I demonstrate to my students the universal acceptability of the basic logical rules? Suppose I have to teach them *Modus Ponens* ($p \rightarrow q$, p; $\therefore$ q) and *Modus Tolens* ($p \rightarrow q$, $\sim q$; $\therefore \sim p$). Most of them seem to understand those. But what they do not understand is what do those rules *look like*? I normally do two things.

I exemplify the rules and compare them with their stupid counterparts namely ($p \rightarrow q$, q; $\therefore p$) and ($p \rightarrow q$, $\sim p$; $\therefore \sim q$) respectively. The students do immediately realize that ($p \rightarrow q$, q; $\therefore p$) is not a rule. But many of them think that ($p \rightarrow q$, $\sim p$; $\therefore \sim q$) is a rule. They argue: "If it rains then the soil gets wet; it is not raining; therefore the soil is not getting wet. Yes, this is fine". I tell them, "Maybe the soil is wet because you are pouring water on the ground. It need not rain". On the other hand, just see: if it is true that "If it rains then soil gets wet", then it can't rain when the soil is not wet". Normally it works. Little by little, students start appreciating the difference between a rule and a non-rule.

By extending the same technique I teach them the rules of Existential Instantiation (EI) and Universal Generalization (UG) through a demonstration of fallacies. Here I'm discussing the Kalish-Montague version of the rules (1964, pp. 118-122). First the UG rule. Suppose I want to prove that ANY triangle x is such that the sum of its angles is 180°. Here I must make sure that x is a randomly chosen triangle about which I know just one thing: it has the property of being a triangle. If I already know that it is an equilateral one, then I shall end up proving that "ANY equilateral triangle x is such that the sum of its angles is 180°", which is definitely not the thing I wanted to prove originally. That means, when I want to prove that "for any x, if x is a triangle [Tx] then it has the property of having three angles whose sum is 180° [Sx]" or $\forall x$ (T$x \rightarrow$ Sx), I have to just that "if x is T then x is S [T$x \rightarrow$ Sx]", and I must not know anything about x before I prove that T$x \rightarrow$ Sx. This has been translated into the technical language of logic in the following way: Prove Fx for proving $\forall x$Fx and see that x never occurs freely before proving Fx. The underlined part is the restriction on the rule. The point here is the following: When I state that $\forall x$Fx or $\exists x$Fx, I mean to say that "everything is F" or "something is F" respectively. In my interpretation there is no space for a variable. But when I state "Fx", I am saying that "x is F". This x, which appears in just "Fx" is a real variable whereas the x, which appears under the scope of a quantifier (as in $\forall x$Fx or $\exists x$Fx), is a pseudo-variable. Note: in the following formula, the first three occurrences of x are within the scope of the quantifier "$\forall$" and, hence are bound, while the last occurrence is free: $\forall x$ (F$x \wedge$ Gx) $\vee$ F$\underline{x}$ [the free occurrence is underlined]. It is clear by now that only a free variable is a real one. Thus, I state that "x is an individual such that the predicate F is true of x" only when I say just "Fx".

Let us see what may happen when we violate the restriction on UG. Suppose my premise is $\exists x$Gx (which may be interpreted at "something is good"). From that, we may show $\forall x$Gx ("everything is good"). The violation of the restriction will allow us to draw this unwelcome conclusion:

Problem I

1. $\exists x Gx$
2. ~~Show~~ $\forall y Gy$
3. Gy — 1, EI [The EI variable must be a new one. It cannot be x.]
4. ~~Show~~ Gy
5. Gy — 3, Repetition

The variable of UG, i.e., y occurs free in line 3 which is before "Show Gy".

Roughly the EI rule is the following: $\exists x Gx$. ∴ Gy [where y is a new variable].

The idea is to express something along these lines: "I know that there is at least one good thing [i.e., $\exists x Gx$; interpret Ga as "a is good"]. From that point, when I conclude that y is good, I'm just naming a good individual "y".

Hence, "y" must be a new variable because I should not know anything else about y. Suppose no such restriction is imposed on EI. And we know that By and $\exists x Gx$. From $\exists x Gx$ we may conclude that Gy. Now I can say that "y is B and G [$By \wedge Gy$]", from which we conclude that "there exists some x such that it is both B and G [$\exists x (Bx \wedge Gx)$]". But my premises just say that "some individual y is bad [By]" and "there exists at least one individual which is good [$\exists x Gx$]". They do not say that "there is at least one individual x which is at the same time bad and good [$\exists x (Bx \wedge Gx)$]". My conclusion cannot claim anything more than what the premises state. This overstatement is due to the violation of the restriction. Even "Gx" cannot be drawn from $\exists x Gx$. Consider the following wrong derivation:

Problem II

1. $\exists x Gx$
2. *Show* $\forall x Gx$
3. *Show* Gx
4. Gx — 1, EI [The restriction is violated, since x is not a new variable.]

The purpose of both the restrictions is to prevent x from being the variable of Universal Generalization when x is used as the variable of Existential Instantiation. In Problem II, x in line 4 comes as a result of EI applied on line 1. And the same x is the variable of UG in line 2. This is undesired. Allowing this would illicitly generalize every specific case. It is evident that without any restriction both of the subsequent derivations can take place anywhere in a proof:

UI: $\forall x Gx$; $\therefore$ Gx or Gy, and **EG:** Gx; $\therefore \exists x Gx$ or $\exists y Gy$.

If $\forall x Gx$, i.e., "everything is G" is true then "x is G" or "y is G". Any variable can fill in the blank in "_ is G". If Gx, i.e., "x is G" is true, then it must be true that "something is G" [$\exists x Gx$]. It does not matter which variable appears in the symbolic formula as long as the former is bound.

Translations

It is often said that the process of translating ordinary sentences into Predicate Logic (PL) is a lot more complicated than into Propositional Logic. Let us discuss a few translation issues that may trouble learners in the beginning phases. We know that "all humans are mortal" is translated as $\forall x\ (Hx \rightarrow Mx)$ and "some human is mortal" as $\exists x\ (Hx \wedge Mx)$. This is taught with the assumption that everybody understands the translations. But the assumption is not always right. Many learners do not understand why these should be translated this way only.

According to PL, "all H are M" means that "for every x, if it is H then it is M [$\forall x\ (Hx \rightarrow Mx)$]". You choose anything; if it is not a human (if it is a dog for instance) then you do not have to check any further. If it is a human, then it must be mortal. That means this sentence does not allow any non-mortal human. Funny enough! This translation has no issues with a world that has absolutely no humans; for it says that if there is a human… etc. On the other hand, "some H is M" means "there exists at least one x, such that it is H and M". You check the things that exist. If at least one of them is both H and M, then this sentence is true. That means this sentence allows non-mortal humans. It is not compatible with a world in which there is no H or M, because it asserts that 1) both the set of humans and the set of mortals are non-empty, and 2) their intersection too is non-empty, i.e., they have at least one common member. We shall see what happens if we translate these sentences differently.

Suppose we do the following: "some human is mortal" = $\exists x\ (Hx \rightarrow Mx)$. This is wrong because the translation says that there exists some x such that if it is a human then it is mortal. That means, the translation has no problems if there are no humans. It is compatible with both the world that has no humans and the world that has some mortal humans as well. But the sentence "some human is mortal" claims that there must be at least one mortal human. It is not compatible with the world without humans. So this translation is wrong. Now let us consider, "all humans are mortal" = $\forall x$ (H$x \wedge$ Mx). This is wrong because the translation makes everything a mortal human; there is space for nothing else. What is wrong with "all humans are mortal" = $\exists x$ (H$x \rightarrow$ Mx)? The sentence, according to PL, says that "if something is a human then it is mortal". The translation too seems to say something similar, i.e., there exists some x such that if it is a human then it is mortal. i.e., no non-mortal human is entertained by the translation. Then what is wrong? The problem is $\exists x$ (H$x \rightarrow$ Mx) is happy when there is at least one individual such that if it is a human then it is mortal. It imposes no upper limit on the number of x. If all humans are mortal, then that is also right. But at the same time, it allows non-mortal humans too, since $\exists x$ (H$x \rightarrow$ Mx) is satisfied even with one mortal human in the world in which both the set of humans and the set of mortal beings are non-empty. It does not mind if all other humans are immortal in that very world. So it does not translate "all humans are mortal". The following table describes the worlds associated with propositions and their proper or improper symbolic representations:

All H are M	Some H is M	$\forall x$(H$x \rightarrow$ Mx)	$\exists x$(H$x \wedge$ Mx)	$\forall x$(H$x \wedge$ Mx)	$\exists x$(H$x \rightarrow$ Mx)
Satisfied if every H is M	Satisfied if there is something which is both H and M	Satisfied if every H is M	Satisfied if there is something which is both H and M	Satisfied if everything is both H and M	Satisfied if there is at least one mortal human when the set of H is non-empty
Does not allow any non-M H	Allows non-M H	Does not allow any non-M H	Allows non-M H	Does not allow any non-M H or non-H M	Allows non-M H
Allows the empty set of H	Does not allow the empty set of H or M	Allows the empty set of H	Does not allow the empty set of H or M	Does not allow the empty set of H or M	Allows the empty set of H

Epilogue

The system-specific logical interpretations of natural-language sentences may not always seem to be intuitively proper. But we must understand that logic is a model of the operations of our intelligence. Like any other model, the model of logic must be compact. The stoic logic represents Propositional Calculus well. I personally prefer Aristotelian Logic to modern Predicate Logic, for I think that

Aristotle's approach is more intuitive than that of PL. For example, when somebody says "all humans are mortal", they normally mean that both the set of humans and the set of mortal beings are non-empty and the former is a subset of the latter. If you ask the speaker whether their universe of discourse can have the empty set of humans, they will probably say *No*. In sum, there is no implication in their sentence. But in the predicate logical interpretation, there is an implication. Thus both "all unicorns are white" and "all humans are mortal" are interpreted in the same way [$\forall x\ (Ux \rightarrow Wx)$ and $\forall x\ (Hx \rightarrow Mx)$ respectively], and share the same truth-value "True". But no naïve speaker would perhaps say that they are equally true. The unicorn-sentence is true because the set of unicorns is empty. Thus there are gaps between our intuitive grasp and logical representations. But, for the sake of compactness, one may accept the modern mathematical model in which you have propositional axioms and rules at the basic level; you add a few more axioms and rules and you get predicate calculus. With some additional rules, this will give you Modal Calculus, and so on. Thus you have a grand system of logic corresponding to the unitary mind that does logical calculations of several types.

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