Acta Mexicana de
Ciencia y Tecnología
Vol. VIII. Nos. 1-2-3. 1974

On a structure defined by A (1.1) Tensor field ξ ($\xi \neq 0, I$) satisfying $\xi^{K} - \xi^{K-2} = 0$.

V. C. Gupta *
(Recieved, july 1974)

SUMMARY

In this paper we have considered the structure defined by a tensor field ξ ($\xi \neq 0$, I) of the type (1.1) satisfying $\xi^{K} - \xi^{K-2} = 0$. In section 1, we have taken k odd and defined the operators l^*, m^* and deduced some results in terms of these operators. Some results are as follows:

$$\xi^{K-2} l^* = l^* \xi^{K-2} = \xi^{K-2}, \quad \xi^{K-2} m^* = m^* \xi^{K-2} = 0.$$

$$pl^* = \xi^{K-2}, \; pm^* = m^*, \quad vl^* = -\xi^{K-2}, \; vm^*, = m^*.$$

where p and q are tensors.

In section 2, we have taken k even and most of the results are similar to those of section 1.

1. Let M^n be an n-dimensional differentiable manifold of class C^∞ and let there be given a tensor field ξ ($\xi \neq o, I$) of the type (1.1) and of class C^∞, satisfying:

$$\xi^{K} - \xi^{K-2} = o. \tag{1.1}$$

where (2 rank ξ $-$ rank ξ^{K-1}) = dim M^n.†

Let us define the operators l^* m^* by

$$l^* = \xi^{K-1}, \; m^* = (I - \xi^{K-1}),\dagger \tag{1.2}$$

I denoting the identity operator. Then we have.

Theorem (1.1): For a tensor field ξ ($\xi \neq 0$, I) satisfying (1.1) the operators l^*, m^* defined by (1.2) and applied to the tangent space at a point of the manifold are complementary projection operators.

Lucknow University, Lucknow, India.
† In section 1 we have taken k odd.

Proof: By virtue of (1.1) and (1.2) we have

$$l^* + m^* = \xi^{K-1} + I - \xi^{K-1} = I$$

$$l^{*2} = \xi^{2K-2} = \xi^{K}.\, \xi^{K-2}$$

$$= \xi^{K-2} \,.\, \xi^{K-2} = \xi^{K}.\, \xi^{K-4}.$$

$$= \xi^{K-2}.\, \xi^{K-4} = \xi^{K}.\, \xi^{K-6}$$

$$= \xi^{K-2}.\, \xi^{K-(K-1)} = \xi^{K-2}.\, \xi$$

$$= \xi^{K-1} = l^*$$

$$m^{*2} = I + \xi^{2K-2} - 2\,\xi^{K-1} = I + \xi^{K-1} - 2\,\xi^{K-1}$$

$$= I - \xi^{K-1} = m^*$$

$$l^*\, m^* = \xi^{K-1}\,(I - \xi^{K-1}) = \xi^{K-1} - \xi^{2K-2}$$

$$= \xi^{K-1} - \xi^{K-1} = 0 = m^*\, l^*$$

which proves the theorem.

Let **L*** and M^* be the complementary distributions corresponding to the projection operators l^*, m^* respectively.

Let the rank of ξ be constant and be equal to r, then from (1.1) we have

$$dim\, L^* = 2r - n \text{ and } dim\, M^* = (2n - 2r).$$

Here $dim\ M^*$ is even but $dim\ L^*$ is not necessarily even. Obviously

$$n \leq 2r \leq 2n.$$

We call such a structure a $\xi(k - (k\text{-}2))$ structure of rank r and the manifold M^n with this structure a $\xi(k, - (k\text{-}2))$ manifold.

Theorem (1.2): For a tensor field $\xi(\xi \neq o, I)$ satisfying (1.1) and l^*, m^* defined by (1.2) we have

$$\xi^{K-2}\, l^* = l^*\, \xi^{K-2} = \xi^{K-2},\ \xi^{K-2}\, m^* = m^*\, \xi^{K-2} = 0.$$

$$\xi^{K-1}\, l^* = l^*,\ \xi^{K-1}\, m^* = 0. \tag{1.3}$$

Proof: By virtue of (1.1) and (1.2), we have

$$\xi^{K-2}\, l^* = \xi^{K-2}\,(\xi^{K-1}) = \xi^{2K-3}$$

$$= \xi^{K}.\, \xi^{K-3} = \xi^{K-2}.\, \xi^{K-3}$$

$$= \xi^{K}.\ \xi^{K-5} = \xi^{K-2}.\ \xi^{K-5}.$$

$$= \xi^{K}.\ \xi^{K-K} = \xi^{K-2}.\ \xi^{K-K}.$$
$$= \xi^{K-2} = l^{*}\ \xi^{K-2}.$$
$$\xi^{K-1}\ l^{*} = \xi(\xi^{K-2}\ l^{*}) = \xi(\xi^{K-2}) = \xi^{K-1} = l^{*}.$$
$$\xi^{K-2}\ m^{*} = \xi^{K-2}(I - \xi^{K-1}) = \xi^{K-2} - \xi^{2K-3}$$
$$= \xi^{K-2} - \xi^{K-2} = 0 = m^{*}\ \xi^{K-2}.$$
$$\xi^{K-1}\ m^{*} = \xi(\xi^{K-2}\ m^{*}) = 0.$$

Corollary 1.1) : ξ acts on L^{*} as an almost product structure operator.

PROOF: By virtue of (1.1) and (1.2) we have

$$\xi^{2}\ l^{*} = \xi(\xi\ l^{*}) = \xi(\xi^{K-2}) = \xi^{K-1} = l^{*}.\ \ \xi\ l^{*} = l^{*}\ \xi = \xi(\xi^{K-1}) = \xi^{K} = \xi^{K-2}.$$

Hence the result.
Corollary (1.2) : The $\xi(k, -(k\text{-}2))$ structure of maximal rank is an almost product structure.

PROOF: If the rank of $\xi = n$, then $dim\ L^{*} = n$ and $dim\ M^{*} = 0$.

In this case $m^{*} = 0$ and $l^{*} = \text{I}$.
Hence ξ satisfies

$$I = \xi^{K-1} = 0.$$

Applying ξ twice to this equation, we get

$$\xi^{2} - \xi^{K+1} = 0 \quad \text{or} \quad \xi^{2} - \xi\ (\xi^{K}) = 0$$

i.e.

$$\xi^{2} - \xi(\xi^{K-2}) = 0 \quad \text{or} \quad \xi^{2} - \xi^{K-1} = 0$$

in consequence of (1.1). Since

$$I - \xi^{K-1} = 0$$

we get

$$\xi^{2} - I = 0.$$

Hence the result.

THEOREM (1.3) : If in M^{n}, there is given a tensor field ξ, $\xi \neq o$ and $\xi^{K-1} \neq I$, of class C^{∞} such that $\xi^{K} - \xi^{K-2} = o$, then M^{n} admits an almost product structure $\zeta = 2\ \xi^{K-1} = I$ where $\zeta = l^{*} - m^{*}$.

$$\xi = l^* - m^* = \xi^{K-1} - (I - \xi^{K-1})$$
$$= 2\,\xi^{K-1} - I.$$

Then $\zeta \neq I$ if $\xi^{k-1} \neq I$. For if possible, suppose $\zeta = I$, then $\xi^{k-1} = I$, contrary to the hypothesis.

Also

$$\zeta^2 = 4\,\xi^{2k-2} + I - 2{\cdot}2\,\xi^{k-1}$$
$$= 4\,\xi^{k-1} + I - 4\,\xi^{k-1} = I.$$

Thus $$\zeta \neq I,\ \zeta^2 = I \text{ if } \xi^{k-1} \neq I.$$

Hence the result.

THEOREM (1.4): Let $\overset{p}{h}$ and q be tensors defined by

$$p = (m^* + \xi^{K-2}),\ q = (m^* - \xi^{K-2}) \tag{1.4}$$

and l^*, m^* be defined by (1.2). We have

$$pl^* = \xi^{K-2},\ pm^* = m^*,\ ql^* = -\,\xi^{K-2},\ qm^* = m^*.$$
$$p^2l^* = l^*,\ p^2m^* = m^*,\ q^2l^* = l^*,\ q^2m^* = m^*. \tag{1.5}$$

i.e. p and q act on L^* as an almost product structure operator and on M^* as an identity operator.

PROOF: In consequence of (1.1), (1.2), (1.3) and (1.4), we have

$$pl^* = (m^* + \xi^{k-2})\,l^* = m^*\,l^* + \xi^{k-2}\,l^* = 0 + \xi^{k-2}\,l^* = \xi^{k-2}.$$
$$p^2l^* = p(pl^*) = (m^* + \xi^{k-2})\,\xi^{k-2} = m^*\,\xi^{k-2} + \xi^{2k-4}$$
$$= 0 + \xi^k.\,\xi^{k-4} = \xi^{k-2}.\,\xi^{k-4}$$
$$= \xi^k.\,\xi^{k-6} = \xi^{k-2}.\,\xi^{k-6}$$

$$= \xi^k.\,\xi^{k-(k-1)} = \xi^{k-2}.\,\xi$$
$$= \xi^{k-1} = l^*.$$
$$pm^* = (m^* + \xi^{k-2})\,m^* = m^{*2} + \xi^{k-2}\,m^* = m^* + 0 = m^*.$$
$$p^2m^* = p(pm^*) = p(m^*) = m^*.$$
$$ql^* = (m^* - \xi^{k-2})\,l^* = m^*l^* - \xi^{k-2}l^*$$
$$= 0 - \xi^{k-2}l^* = -\,\xi^{k-2}.$$

$$q^2 l^* = q(ql^*) = (m^* - \xi^{k-2})(-\xi^{k-2}) = -m^* \xi^{k-2} + \xi^{2k-4}.$$

$$= 0 + \xi^k . \xi^{k-4} = \xi^{k-2} . \xi^{k-4}$$

$$= \xi^k . \xi^{k-6} = \xi^{k-2} . \xi^{k-6}$$

$$= \xi^k . \xi^{k-(k-1)} = \xi^{k-2} . \xi$$

$$= \xi^{k-1} = l^*.$$

$$qm^* = (m^* - \xi^{k-2})\, m^* = m^{*2} - \xi^{k-2} m^* = m^* - 0 = m^*$$

$$q^2 m^* = q(qm^*) = q(m^*) = m^*$$

Hence the result.

THEOREM (1.6): Let

$$\bar{p} = \frac{\xi^{k-1} + \xi^{k-2}}{\sqrt{2}}, \quad \bar{q} = \frac{\xi^{k-1} - \xi^{k-2}}{\sqrt{2}}. \tag{1.6}$$

Then

$$\bar{p}\bar{q} = \bar{q}\bar{p} = 0, \quad \bar{p}^2 - \sqrt{2}\bar{p} = 0, \quad \bar{q}^2 - \sqrt{2}\bar{q} = 0. \tag{1.7}$$

Hence

$$\bar{p}^2 + \bar{q}^2 = \sqrt{2}(\bar{p} + \bar{q}), \quad \bar{p}^2 - \bar{q}^2 = \sqrt{2}(\bar{p} - \bar{q}). \tag{1.8}$$

PROOF: By virtue of (1.6) and (1.1), we have

$$\bar{p}\bar{q} = \bar{q}\bar{p} = \tfrac{1}{2}[\xi^{2k-2} - \xi^{2k-4}] = \tfrac{1}{2}[\xi^k . \xi^{k-2} - \xi^{2k-4}]$$

$$= \tfrac{1}{2}[\xi^{k-2} . \xi^{k-2} - \xi^{2k-4}] = \tfrac{1}{2}[\xi^{2k-4} - \xi^{2k-4}] = 0.$$

$$\bar{p}^2 = \tfrac{1}{2}[\xi^{2k-2} + \xi^{2k-4} + 2\,\xi^{2k-3}],$$

$$\bar{q}^2 = \tfrac{1}{2}[\xi^{2k-2} + \xi^{2k-4} - 2\,\xi^{2k-3}].$$

Now

$$\xi^{2k-3} = \xi^k . \xi^{k-3} = \xi^{k-2} . \xi^{k-3}$$

$$= \xi^k . \xi^{k-5} = \xi^{k-2} . \xi^{k-5}.$$

$$= \xi^k . \xi^{k-k} = \xi^{k-2} . \xi^{k-k}$$

$$= \xi^{k-2} . \xi^{2k-2} = \xi^k . \xi^{k-2} = \xi^{k-2}\, \xi^{k-2} = \xi^{2k-4} . \xi^{2k-2}$$

$$= \xi^{2k-4} = \xi^{k-1}.$$

Hence

$$\bar{p}^2 = \tfrac{1}{2}[2\,\xi^{K-1} + 2\,\xi^{K-2}] = \xi^{k-1} + \xi^{k-2} = \sqrt{2}\bar{p}.$$

and

$$\bar{q}^2 = \tfrac{1}{2}[2\,\xi^{k-1} - 2\,\xi^{k-2}] = \xi^{k-1} - \xi^{k-2} = \sqrt{2}\bar{q}.$$

Consequently

$$\bar{p}^2 - \bar{q}^2 = \sqrt{2}(\bar{p} - \bar{q}).$$

and

$$\bar{p}^2 + \bar{q}^2 = \sqrt{2}(\bar{p} + \bar{q}).$$

2. Let M^n be an n dimensional differentiable manifold of class C^∞ and let there be given a tensor field $\xi(\xi \neq o, I)$ of type (1.1) and of class C^∞, satisfying

$$\xi^k - \xi^{k-2} = o \tag{2.1}$$

where (2 rank ξ − rank ξ^{k-2}) $= dim\ M^n$.†

Let us define the operators l^*, m^* by

$$l^* = \xi^{k-2},\ m^* = (I - \xi^{k-2}),\dagger \tag{2.2}$$

I denoting the identity operator. Then we have

Theorem (2.1). For a tensor field $\xi(\xi \neq o, I)$ satisfying (2.1), the operators l^*, m^* defined by (2.2) and applied to the tangent space at a point of the manifold are complementary projection operators.

Proof: The proof is similar to that of theorem (1.1).

Let L^* and M^* be the complementary distributions corresponding to the projection operators l^*, m^* respectively.

Let the rank of ξ be constant and be equal to r, then

$$dim\ L^* = (2r - n) \text{ an } dim\ M^* = (2n - 2r).$$

Here the dim M^* is given but dim L^* is not necessarily even. Obviously, $n \leq 2r \leq 2n$.

We call such a structure a $\xi(k, -(k\text{-}2))$ structure of rank r and the manifold M^n with this structure a $\xi(k, -(k\text{-}2))$ *manifold.*

We shall now state the following theorems. The proofs following a manner as as in the previous section.

† In section 2, we have taken k even.

THEOREM (2.2): For a tensor field $\xi(\xi \neq o, I)$ satisfying (2.1) and l^*, m^* defined by (2.2) we have

$$\xi^{k-2}\, l^* = l^*,\ \xi^{k-2}\, m^* = 0.$$

$$\xi^{k-1}\, l^* = \xi^{k-1},\ \xi^{k-1}\, m^* = o. \tag{2.3}$$

Corollary (2.1): ξ acts on L^* as an almost product structure operator.
Corollary (2.2): The $\xi(k, -(k\text{-}2))$ structure of maximal rank is an almost product structure.

THEOREM (2.3): If in M^n, there is given a tensor field ξ, $\xi \neq o$ and $\xi^{k-2} \neq I$, of class C^∞ and such that $\xi^k - \xi^{k-2} = o$, then M^n admits an almost product structure $\zeta = 2\xi^{k-2} - I$, where $\zeta = l^* - m^*$.

THEOREM (2.4): Let p and q be tensors defined by

$$p = (m^* + \xi^{k-1}),\ q = (m^* - \xi^{k-1}) \tag{2.4}$$

and l^*, m^* be defined by (2.2). We have

$$pl^* = \xi^{k-1},\ pm^* = m^*,\ ql^* = -\xi^{k-1},\ qm^* = m^*.$$

$$p^2 l^* = l^*,\ p^2 m^* = m^*,\ q^2 l^* = l^*,\ q^2 m^* = m^*. \tag{2.5}$$

i.e. p and q act on L^* as an almost product structure operator and on M^* as an identity operator.

THEOREM (2.5): Let

$$\bar{p} = \frac{\xi^{k-1} + \xi^{k-2}}{\sqrt{2}},\ \bar{q} = \frac{\xi^{k-1} - \xi^{k-2}}{\sqrt{2}}. \tag{2.6}$$

Then

$$\bar{p}\,\bar{q} = \bar{q}\,\bar{p} = 0,\ \bar{p}^2 - \sqrt{2}\,\bar{p} = 0,\ \bar{q}^2 + \sqrt{2}\,\bar{p} = 0. \tag{2.7}$$

Hence

$$\bar{p}^2 + \bar{q}^2 = \sqrt{2}(\bar{p} - \bar{q}),\ \bar{p}^2 - \bar{q}^2 = \sqrt{2}(\bar{p} + \bar{q}). \tag{2.8}$$

ACKNOWLEDGEMENTS

For his kind help and tender guidance I am very much indebted to Dr. M. D. Upadhyay.

REFERENCES

1. Mishra, R. S.: A course in Tensors with applications to Riemannian Geometry. Pothishala, Pvt. Ltd. Allahabad (1965).
2. Yano, K.: Differential Geometry on Complex and almost Complex spaces. Pergamon Press New York (1965).
3. Mishra, R. S.: On almost product and almost decomposable manifold Tensor N. S. Vol. 21, p. 255-60 (1970).