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On a generalization of Laplace transform

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INTRODUCTION

The aim of this note is to generalize the classical Laplace Transform and its inversion formula in a new manner. Recently the generalization of well known Laplace Transform

$$\Phi(p) = p \int_0^\infty e^{-px} f(x)\, dx \tag{1.1}$$

has been given by Gupta [4] in the form

$$\Phi(p) = \alpha \int_0^\infty (px)^{\rho-1} E\mu(\alpha p x)\; H^{k_3,k_4}_{k_1,k_2}\left[z(px)^\sigma \middle| \begin{matrix} \{(a_{k}, e_{k})\} \\ \{(b_{k_2}, \beta_{k_2})\} \end{matrix} \right] f(x)\, dx, \tag{1.2}$$

where $E\mu(x)$ is exponential function defined by Busbridge [1] and

$$H^{m,n}_{p,q}\left[x \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \alpha_q)\} \end{matrix} \right]$$

is H-function defined by Fox C.[3] $\{(a_p, \alpha_p)\}$ denotes the set of parameters $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$. We put a new generalization of (1.2) in the form

$$\Phi(p) = \alpha \int_0^\infty (px)^{\rho-1}\, G^{m_1,0}_{r_1,t_1}\left[\alpha p x \middle| \begin{matrix} \{(a_{r_1})\} \\ \{(b_{t_1})\} \end{matrix} \right] \times$$

$$\times\; H^{m_2,n_2}_{r_2,t_2}\left[z(px)^\sigma \middle| \begin{matrix} \{(e_{r_2}, \epsilon_{r_2})\} \\ \{(f_{t_2}, T_{t_2})\} \end{matrix} \right] f(x)\, dx \tag{1.3}$$

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where $\{(a_r)\}$ denotes the set of parameters $a_1, a_2, \ldots, a_n$ and

$$G_{p,q}^{m,n}\left[x \middle| \begin{matrix} \{(a_r)\} \\ \{(b_q)\} \end{matrix} \right]$$

is G-function defined by Meijer.[2]

The transform (1.3) exists under the following set of conditions

$$\alpha>0,\ z>0,\ \sigma>0,\ 0\leq m_1\leq t_1,\ 0\leq r_1\leq t_1,$$

$$2m_1> t_1 + r_1,\ 0 \leq m_2 \leq t_2,\ 0 \leq n_2 \leq r_2.$$

$$\sum_{j=1}^{n_2} \epsilon_j - \sum_{j=1+n_2}^{r_2} \epsilon_j + \sum_{j=1}^{m_2} T_j - \sum_{j=1+m_2}^{t_2} T_j \equiv \eta>0$$

$$\eta' = m_1 - \frac{r_1}{2} - \frac{t_1}{2} > 0,\ |arg\, \alpha p| < \Pi\eta'$$

$$|arg\, z\, p^\sigma| < \frac{\Pi\eta}{2},\ x^{\rho+\rho_1+\rho_2-1} f(x) \in L(O,R)$$

$$\rho_1 = min\, (b_i), i = 1, 2, \ldots, m_1$$

$$\rho_1 = min\, (\sigma\frac{f_h}{h}), h = 1, 2, \ldots, m_2$$

PARTICULAR CASES:

Taking $r_1 = 1$, $m_1 = t_1 = 2$, (1.3) reduces to (1.2). Further if we put $\rho = 2$, $\mu = k_1 = 0 = k_4$, $k_2 = k_3 = 1 = e_{k1} = f_{k2}$, $\alpha_{k3} = b_{k1} = 0$ $\alpha + z = 1$ it reduces to (1.1).

We shall use the following result of Kalia [5] for finding the inversión of the transform (1.3)

$$\int_0^\infty t^{\rho-1} G_{r,l}^{k,f}\left[st \middle| \begin{matrix} \{(c_r)\} \\ \{(d_l)\} \end{matrix} \right] H_{p,q}^{m,n}\left[zt^\sigma \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] dx =$$

$$= s^{-\rho} H_{p+l,q+r}^{m+f,k+n}\left[\frac{z}{s^\sigma} \middle| \begin{matrix} \{(a_n,\alpha_n)\}, \{(1 - d_l - \rho,\sigma)\}, \\ (a_{n+1},\alpha_{n+1}),\ldots,(a_p,\alpha_p), \\ \{(b_{m},_{m})\}, \{(1-c_r-\rho,\sigma)\}, \\ (b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q). \end{matrix} \right] \quad (1.4)$$

provided

(i) $R\left(\rho + \sigma\frac{b_h}{B_h} + d_i\right) > 0,\ (h = 1,2,\ldots,m),\ (i = 1,2,\ldots,k),\ \sigma > 0$

(ii) $R\left(\rho + (c_j - 1) + \sigma(\frac{a_{h'} - 1}{\alpha_{h'}})\right) > 0,\ (j = 1,2,\ldots,f),\ (h' = 1,2,\ldots,n)$

(iii) $\sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=1+m}^{q} \beta_j \equiv \lambda > 0,\ 2k + 2f - l - r \equiv \mu > 0$

$|\arg z| < \frac{\Pi\lambda}{2},\ |\arg s| < \frac{\mu\Pi}{2}$

The following theorem will give the inversion of the transform (1.3).

THEOREM: If

$$h(x) = \frac{1}{2\Pi i}\int_{u-1\infty}^{u+i\infty} \frac{x^{-s}\,ds}{H(s)}, \tag{1.5}$$

$$H(s) = \alpha^{s-\rho-1}\ H^{m_2,n_2+m_1}_{r_2+t_1,t_2+r_1}\left[\frac{z}{\alpha\sigma}\ \middle|\ \begin{matrix}\{(e_{n_2}, \in_{n_2})\}, \{(1 - b_{t_1} - \varrho + s, \sigma)\}, \\ (e_{n_2+1}, \in_{n_2+1}), \ldots, (e_{r_2}, \in_{r_2}) \\ \{(f_{m_2}, T_{m_2})\}, \{(1 - a_{r_1} - \varrho + s, \sigma)\}, \\ (f_{m_2+1}, T_{m_2+1}), \ldots, (f_{t_2}, T_{t_2})\end{matrix}\right]$$

then

$$f(x) = \int_0^\infty h(px)\,\phi(p)\,dp \tag{1.6}$$

where $\phi(p)$ is given by (1.3) provided $|h(x)|$ exists and (1.6) is absolutely convergent, $p^{-u}\,\phi(p) \in L\,(0, \infty)$, $x^{u-1}\,f(x) \in L\,(0, \infty)$, $R\,(\varrho_1 + \varrho_2 + \varrho) > u$.

PROOF: Substituting the value of $\phi(p)$ from (1.2) we get

$$\int_0^\infty p^{-s}\,\phi(p)\,dp = \int_0^\infty p^{-s}\left\{\alpha\int_0^\infty (px)^{-\rho_1}\ G^{m_1,0}_{r_1,t_1}\left[\alpha px\ \middle|\ \begin{matrix}\{(a_{r_1})\} \\ \{(b_{t_1})\}\end{matrix}\right]\right.$$

$$\left. H^{m_2,n_2}_{r_2,t_2}\left[z(px)\,\sigma\ \middle|\ \begin{matrix}\{(e_{r_2}, \in_{r_2})\} \\ \{(f_{t_2}, T_{t_2})\}\end{matrix}\right] f(x)\,dx\right\}\ dx$$

Due to absolute convergence of the integrals involved we have on inversion of the integrals and simple substitution

$$\int_0^\infty p^{-s}\,\phi(p)\,dp = \alpha \int_0^\infty x^{s-1} f(x)\,dx \int_0^\infty v^{\rho-s-1}\, G^{m_1,0}_{r_1,t_1}\left[\alpha v \,\middle|\, \begin{matrix} \{(a_{r_1})\} \\ \{(b_{t_1})\} \end{matrix}\right]$$

$$H^{m_2,n_2}_{r_2,t_2}\left[z v^{\sigma} \,\middle|\, \begin{matrix} \{(e_{r_2}, \epsilon_{r_2})\} \\ \{(f_{t_2}, T_{t_2})\} \end{matrix}\right] dv$$

Evaluating the v integral with the help of (1.5) we have

$$\int_0^\infty p^{-s}\,\phi(p)\,dp = H\ (s) \int_0^\infty x^{s-1}\,dx$$

Applying the Mellin's Inversion formula, we get the result (1.6).

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