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Application of Gamma functions in solving certain integral equations

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ABSTRACT

An integral equation involving the G — function[1] (p. 207) as kernel has been transformed, by using the technique of Gamma functions, into another integral equation with symmetrical Fourier kernel[3] and the solution is then immediate. Later some special cases are discussed.

1. INTRODUCTION

In the present note we have employed the Γ operator in introducing new Gamma function factors into the integrand for transforming the given integral equation into another integral equation involving a symmetrical Fourier kernel and the required solution is then immediate consequence. The present Fourier kernel is a generalisation of a large variety of functions occurring in various branches of science.

We shall find the solution of the following type of integral equation:

$$\int_0^\infty G_{2p,q}^{0,p}\left(xu \,\middle|\, \begin{matrix}(a_i)_{2p}\\(b_i)_q\end{matrix}\right) f(u)\,du = \Phi(x),\ (x > 0), \tag{1.1}$$

where Φ is given and f is the function to be found. The given function $\Phi(x)$ is supossed to be an L_2 — function. In (1.1), the Meijer's G — function [1] (p. 207) expressed as:

$$G_{2p,q}^{0,p}(x) = (2\pi i)^{-1} \int_T M_{q,2p}(s)\, x^{-s}\, ds, \tag{1.2}$$

where

$$M_{q,2p}(s) = \prod_1^p \Gamma(a_i - cs)\ \{\prod_1^v \Gamma(b_i + c - cs) \prod_1^p \Gamma(a_i - c + cs)\}^{-1}. \tag{1.3}$$

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We now use the asymptotic expansion of the Gamma function:[6]

(i) $c > 0$, $h = c(q - 2p) > 0$;

(ii) all the poles of the integrand of (1.2) are simple;

(iii) the contour T is a straight line parallel to the imaginary axis in the complex $s -$ plane ($s = \sigma + i\tau$, where σ and τ are real) and the poles of $\Gamma(a_i - cs)$ lie to the right of it.

We now use the asymptotic expansion of the Gamma function[6]:

$$log\ \Gamma(s + a) = (s + a - \tfrac{1}{2})\ log\ s - s + \tfrac{1}{2}\ log(2\pi) + 0(s^{-1}), \qquad (1.4)$$

where $|arg\ s| < \pi$ and 0 is the order symbol, in finding the asymptotic expansion of $M_{q,2p}(s)$, $s = \sigma + i\tau$, σ and τ real, when $|\tau|$ is large. The result is expressed as:

$$.M_{q,2p}(s) = |\tau|^{h(\sigma-\frac{1}{2})}\ exp\{i\tau(h\ log|\tau| - B\}\ \{Q + O(|\tau|)^{-1}\}, \qquad (1.5)$$

where B is a constant and Q is a constant which may have one value when τ is large and negative. From (1.5), it follows that if $\sigma < \frac{1}{2}$ then the integral of (1.2) is uniformly convergent with respect to x. It can be extended to the case $\sigma = \frac{1}{2}$.

2. PRELIMINARY RESULTS

The Gamma function denoted by $\Gamma(z)$ is defined as:

$$\Gamma(z) = \int_0^\infty e^{-x}\, x^{z-1}\, dx, \quad Re(z) > 0. \qquad (2.1)$$

If $M[h(u)] = H(s)$ and $M[f(u)] = F(s)$, the Parseval Theorem states that

$$\int_0^\infty h(u)\ f(u)\ du = (2\pi i)^{-1} \int_C H(s)\ F(1\text{-}s)\ ds, \qquad (2.2)$$

where C is a suitable contour in the $s -$ plane and M denotes the Mellin transform.

3. THE SOLUTION OF (1.1) AS AN INTEGRAL EQUATION

THEOREM. If,

(i) $c > 0$, $Re(b_i) > -c/2$, $i = 1, \ldots, q$;

(ii) $f(x) \in L_2(0, \infty)$;

(iii) $s^{h(s-\frac{1}{2})}\ F(1\text{-}s) \in L(\tfrac{1}{2} - i\,\infty, \tfrac{1}{2} + i\,\infty)$, $h = c(q - 2p) > 0$;

(iv) $F(1\text{-}s) \in L(\frac{1}{2} - i\,\infty, \frac{1}{2} + i\,\infty)$;

(v) $y^{-\frac{1}{2}} f(y) \in L(0, \infty)$, *where* $f(y)$ is of bounded variation near the point $y = x$, then the solution of (1.1), as an integral equation for $f(u)$ is:

$$f(x) = \int_0^\infty G^{q,p}_{2p,2q}(xu).\ .\{u^{b_1}\Gamma_1[t^{b_1-1}\{t^{b_2}\Gamma_2[\ldots\{t^{b_v}\Gamma_v u^{b_v-1}\Phi(u^{-c})]_{t=\frac{1}{\tau_1}}\}]\ldots]\}du. \tag{3.1}$$

Proof. Firstly we apply (2.2) to the left side of (1.1). For large positive u and $x > 0$, the asymptotic expansion of $G^{0,p}_{2p,q}(xu)$ discussed in Fox,[3] and conditions (i) and (iii), allow us to use Theorem 72, p. 95 [5] with $k = \frac{1}{2}$. Thus, we can apply (2.2) to the left—hand side of (1.1). By using (1.3), we obtain

$$\Phi(x) - (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{q,2p}(s)\ x^{-s}\,F(1-s)\ ds,\ x > 0, \tag{3.2}$$

where $M_{q,2p}(s)\ x^{-s}$ and $F(s)$ are Mellin transforms of $G^{0,p}_{2p,q}(xu)$ and $f(u)$ respectively, and the contour in the s-plane is the straight line $s = \frac{1}{2}\ i\tau$, τ varies from $-\infty$ to ∞.

In this section of the proof, we shall try to introduce new Gamma function factors into the integrand of (3.2) by means of operator Γ. In the first instance we introduce q-*th* new Gamma function factor $\Gamma(b_q + cs)$ into the integrand by using the technique of operator Γ. Then in similar fashion we can introduce another $(q-1)$ factors, namely, $\Gamma(b_{q-1} + cs)$, $\Gamma(b_{q-2} + cs)$, $\ldots$, $\Gamma(b_1 + cs)$, and we can arrive at the result with symmetrical Fourier kernel and the formal solution then would be immediate.

Now we use operator Γ_v to introduce the new Gamma function factor $\Gamma(b_q + cs)$, $c > 0$, $Re(b_q) > -c/2$, into the integrand of (3.2).

Considering $\Phi(x^{-c})$, we are led to the following result:

$$\Phi(x^{-c}) = (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{q,2p}(s)\ x^{cs}\,F(1-s)\ ds. \tag{3.3}$$

Using the operator, we find

$$\Gamma_v[x^{b_q-1}\,\Phi(x^{-c})] = \int_0^\infty e^{-tx}\,x^{b_q-1}\{(2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{q,2p}(s)\ x^{cs}\,F(1-s)\ ds\}\ dx. \tag{3.4}$$

Now using (1.5), one can obtain easily for large $|\tau|$, $s = \frac{1}{2} + i\tau$,

$$M_{q,2p}(s)\ F(1-s) = s^{h(s-\frac{1}{2})}\,F(1-s)\ \{Q_1 + 0(s^{-1})\}, \tag{3.5}$$

where Q_1 is a constant which may have one value when τ is large and positive and another value when τ is large and negative.

Since $s = \frac{1}{2} + i\tau$, the real power of x in (3.4) is $Re(b_q) + \frac{c}{2} - 1$, which by condition (i), exceeds -1. Also, by (3.5) and condition (iii), the terms in s in (3.4) belong to $L(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, the integral in (3.4) is an absolutely convergent double integral and consequently we integrate with respect to x. The result, thus, found is:

$$\Gamma_v[x^{b_q-1}\,\Phi(x^{-c})] = (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{q,2p}(s)\;\Gamma(b_q + cs)\; t^{-b_q-cs}\,F(1 - s)\;ds. \tag{3.6}$$

In order to apply again operator Γ_{v-1}, we have to write $t = 1/\tau_1$, in (3.6), thus we find

$$\{\{t^{b_q}\,\Gamma_v[x^{b_q-1}\,\Phi(x^{-c})]\}_{t=\frac{1}{\tau_1}}\} = (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{q,2p}(s)\;\Gamma(b_q + cs)\;t_1^{cs}\,F(1 - s)\;ds. \tag{3.7}$$

Again we can apply operator Γ_{q-1} in (3.7) in order to introduce $\Gamma(b_{q-1} + cs)$ as in the previous case, after justifying the inversion in the order of integration. Thus, by $(q - 1)$ successions of Γ operator, we can arrive at the result:

$$\{x^{b_1}\Gamma_1[t^{b_1-1}\{t^{b_2}\Gamma_2[\ldots\ \{t^{b_v}\,\Gamma_v[x^{b_v-1}\,\Phi(x^{-c})]\}_{t=\frac{1}{\tau_1}}]\ldots]\} \tag{3.8}$$

$$= (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{2q,2p}(s)\;x^{-cs}\,F(1 - s)\;ds.$$

If we write the left—hand side of (3.8) equal to $\Psi(x)$ and replace x by $x^{1/c}$, then (3.8) takes the form

$$\psi(\mathbf{x}) = (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{2q,2p}(s)\;x^{-s}\,F(1 - s)\;ds \tag{3.9}$$

or

$$\psi(\mathbf{x}) = (2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M\left[\;G^{q,p}_{2p,2q}\,(u)\;\right]\;x^{-1}\,F(1 - s)\;ds, \tag{3.10}$$

in the light of (1.3).

On using,[4] p. 391, eqn. (12), to the right—hand side of (3.10), it can be written as an integral involving the product of $G^{q,p}_{2p,2q}(xu)$ and $f(u)$. The result, thus obtained is expressed as:

$$\int_0^\infty G^{q,p}_{2p,2q}(xu)\, f(u)\, du = \psi(x). \tag{3.11}$$

Since $G^{p,q}_{2p,2q}(xu)$ is a symmetrical Fourier kernel, we can write formally the solution as

$$f(x) = \int_0^\infty G^{q,p}_{2p,2q}(xu)\, \psi(u)\, du. \tag{3.12}$$

4. APPLICATIONS

As a great many of special functions occurring in problems of applied mathematics can be reduced from the G—function, one can derive the solutions of several integral equations, by specialising the G—function, from (3.1).

When $c = 1$, $p = 2$, $q = 2$, our theorem leads to the:

Corollary. Under the conditions of the theorem, the following integral equation

$$\int_0^\infty G^{0,2}_{4,2}(xu)\, f(u)\, du = \Phi(x), \quad x > 0, \tag{4.1}$$

has the solution

$$f(x) = \int_0^\infty G^{2,2}_{4,4}(xu)\ .\ u^{b_1}\Gamma_1[t_1^{b_1-1}\{t^{b_2}\ \Gamma_2[u^{b_2-1}\,\Phi(u^{-1})]\}_{t=\frac{1}{\tau_1}}]\, du. \tag{4.2}$$

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