Acta Mexicana de
Ciencia y Tecnología
Vol. VIII. Nos. 1-2-3. 1974

Study of F-structure manifold defined by $f^3 + f = 0$

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(Recieved, december 1973)

SUMMARY

The idea of f-structure manifold on a differentiable manifold was initiated and developed by Yano [2, 4], Koto [3] defined and studied certain structures on almost Hermitian manifold. Later on Gray [5] reformulated certain formula (given by Koto [3]) in terms of co-derivatives and exterior derivatives. This paper is devoted to the study of some structures in terms of exterior, co-derivative and Lie-derivatives.

In section 2, we define certain structures and obtain some theorems relating to these structures. In section 3, we have defined some operators and have established some results on these operators.

1. INTRODUCTION

An n-dimensional differentiable manifold M is said to possess an f-structure if a non-null (1.1) tensor field f of constant rank r is defined on it which satisfies [4]:

$$f^3 + f = 0. \tag{1.1 a}$$

If the rank of f is such that $n - r \geqslant 1$, there exist two complementary distribution L and M corresponding to the projection operators l and m respectively, defined by [4]

$$l = -f^2, \quad m = f^2 + I, \tag{1.1 b}$$

where I denotes the identity operator these projection operators satisfy the following results [4]

$$lf = fl = f, \quad mf = fm = 0,$$
$$f^2 l = -l, \quad f^2 m = 0. \tag{1.2}$$

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Thus from the above relations it is observed that f acts on l as an almost complex structure and on m as a null operator. If the rank of f is r, then the dimension of L and M are r and $(n\text{-}r)$ respectively [4].

Let $F(M)$ denote the ring of real valued differentiable functions on M and $X(M)$ the module of derivations of $F(M)$. $X(M)$ is then a Lie algebra over real numbers and elements of $X(M)$ are called vector fields. The (1.1) tensor field f is then a linear map over $X(M)$, such that

$$f\colon X(M) \longrightarrow X(M)$$

Yano [4] has defined a positive definite Riemannian metric in M, with respect to which the distribution L and M are orthogonal. Such a Riemannian metric satisfies the following relations [4]

$$<X, Y> = <fX, fY> + <mX, Y>, \text{ for all } X, Y \in X\ (M) \tag{1.3}$$

since the distribution L and M are orthogonal, by virtue of (1.2), we have

$$<fX, Y> = <f^2X, fY> \qquad <X, fY> = <fX, f^2X> \tag{1.4}$$

A 2-form F has been defined as [4],

$$F(X,Y) = - F(Y,X) = <fX,Y>. \tag{1.5}$$

By virtue of the definitions of the Riemannian connection and the Lie-derivative L_X, we have the following relations

$$\begin{aligned} &\text{a) } \nabla_X(f)(Y) = \nabla_X(fY) - f\nabla_XY, \\ &\text{b) } (L_Xf)Y = [X, fY] - f[X,Y] \end{aligned} \tag{1.6}$$

From (1.2) and (1.6) a, b we have

$$m(\nabla_Xf)(mY) = 0 \quad \text{and} \quad m(L_Xf)(mY) = 0.$$

Since f^2 is also a (1.1) tensor, we have

$$\nabla_X(f^2)(Y) = \nabla_X(f^2(Y)) - f^2\,\nabla_XY \tag{1.7}$$

The covariant derivative $\nabla_X(F)$ and the exterior derivative dF of F are given by the following relations:

$$\nabla_X(F)(Y,Z) = <\nabla_X(f)(Y), Z>, \tag{1.8}$$

and

$$dF(X,Y,Z) = \sum_{X,Y,Z} \nabla_X(F)(Y,Z), \text{ where } \sum_{X,Y,Z} \tag{1.9}$$

denotes the cyclic sum over X,Y,Z.

For an f-structure manifold, we have

$$\nabla_X(F)(f^2Y, fZ) = \nabla_X(F)(fY, f^2Z) \tag{1.10}$$

and

$$\nabla_X(F)(f^2Y, f^2Z) = -\nabla_X(F)(fY, fZ) \tag{1.11}$$

2. In this section we shall give some definitions and obtain some results.
A f-structure manifold is called
a) f-Kählerian (fK) manifold iff $\nabla_{fX} f = 0$,
b) f-almost Kählerian (fAK) manifold iff

$$dF(fX, fY, fZ) = 0. \tag{2.1}$$

where

$$dF(fX, fY, fZ) = \nabla_{fX}(F)(fY, fZ) + \nabla_{fY}(F)(fZ, fX) + \nabla_{fZ}(F)(fX, fY)$$

c) f-nearly Kählerian (fNK) manifold iff

$$\nabla_{fX}(f)(fY) + \nabla_{fY}(f)(fX) = 0,$$

d) f-Quasi Kählerian (fQK) manifold iff

$$\nabla_{fX}(f)(fY) + \nabla_{f^2X}(f)(fY) = 0,$$

e) f-Hermitian (fH) manifold iff $N(fX, fY, fZ)$ 0, for all $X, Y, Z \in (M)$

Theorem 2.1. The *nasc* for an f-structure manifold to be an f-nearly-Kählerian maniofld is that

$$f\{\nabla_{fX}(f(fY)) + \nabla_{fY}(f(fX))\} + l(\nabla_{fX} fY + \nabla_{fY} fX) = 0 \tag{2.2}$$

Proof. We have

$$\begin{aligned} \nabla_{fX}(f(fY)) + \nabla_{fY}(f(fX) &= (\nabla_{fX} f)(fY) + \\ &+ f\nabla_{fX} fY + (\nabla_{fY} f)(fX) + f\nabla_{fY} fX \end{aligned} \tag{2.3}$$

From (2.1)c and (2.3) we get

$$\nabla_{fX}(f(fY)) + \nabla_{fY}(f(fX)) = + f\{\nabla_{fX} fY + \nabla_{fY} fX\}$$

operating the above expression by f throughout and using (1.1)b we get (2.2). This proves the first part of the theorem. The converse is obvious.

Theorem 2.2. The *nasc* for an f-structure manifold to be an f-Quasi Kählerian manifold is that

$$f\{\nabla_{fX}(f(fY)) + (\nabla_{f^2}(f)(f^2Y))\} = -l\{\nabla_{fX} fY + \nabla_{f^2X} f^2Y\} \tag{2.4}$$

PROOF. We have

$$\nabla_{fX}(f(fY)) + \nabla_{f^2X}(f(f^2Y)) = (\nabla_{fX}f)(fY) + \\ + f\nabla_{fX}fY + (\nabla_{f^2X}f)(f^2Y) + f\nabla_{f^2X}f^2Y \tag{2.5}$$

From (2.1)d, (2.5) and (1.1)b, the proof follows atonce.

THEOREM 2.3. The necessary condition for an f-Quasi Kählerian manifold to be f-Kählerian manifold is that

$$\nabla_{fX}fY + \nabla_{f^2X}f^2Y = 0 \tag{2.6}$$

PROOF. We have for an f-Quasi Kählerian manifold

$$\nabla_{fX}(f)(fY) + \nabla_{f^2X}(f)(f^2Y) = 0$$

from which we get

$$(\nabla_{fX}(f(fY)) - f\nabla_{fX}fY + \nabla_{f^2X}(f(f^2Y)) - f\nabla_{f^2X}f^2Y = 0. \tag{2.7}$$

If we suppose that f-Quasi Kählerian manifold is f-Kählerian. Thus using (2.1)a in (2.7) we get (2.6)

THEOREM 2.4. If an f-structure manifold has any two of the following properties, it has third also.

a) it is f-nearly Kählerian manifold,
b) it is f-Quasi Kählerian manifold
c) it is f-structure manifold for which

$$\nabla_{fY}f(fX) = \nabla_{f^2X}f(f^2Y) \tag{2.8}$$

PROOF. Let us assume that

$$A(X,Y) = \nabla_{fX}f(fY) + \nabla_{fY}f(fX),$$

$$B(X,Y) = \nabla_{fX}f(fY) + \nabla_{f^2X}f(f^2Y)$$

from the above we get

$$A(X,Y) - B(X,Y) = \nabla_{fX}f(fX) - \nabla_{f^2X}f(f^2Y) \tag{2.9}$$

From (2.9) we see that if any two properties hold the third one also holds.

THEOREM 2.5. The condition for an f-structure manifold to be f-almost Kählerian is that

$$\nabla_{f^2X}(F)(f^2Y, fZ) + \nabla_{f^2Y}(F)(f^2Z, fX) + \nabla_{f^2Z}(F)(f^2X, fY) = 0.$$

Proof. Since for an f-almost Kählerian manifold, we have

$$\nabla_{fX}(F)(fY,fZ) + \nabla_{fY}(F)(fZ,fX) + \nabla_{fZ}(F)(fX,fY) = 0$$

which gives us

$$\nabla_{fX}(F)(fY,fZ) = -\nabla_{fX}(F)(fZ,fX) - \nabla_{fZ}(F)(fX,fY)$$

Therefore

$$\begin{aligned}\nabla_{f^2X}(F)(f^2Y,fZ) &= -\nabla_{f^2Y}(F)(fZ,f^2X) - \nabla_{fZ}(F)(f^2X,f^2Y) \\ &= -\nabla_{f^2Y}(F)(fZ,f^2X) + \nabla_{fZ}(F)(fX,fY)\end{aligned} \tag{2.10}$$

Similarly we have

$$\nabla_{f^2Y}(F)(f^2Z\,fX) = -\nabla_{f^2Z}(F)(fX,f^2Y) + \nabla_{fX}(F)(fY,fZ) \tag{2.11}$$

and

$$\nabla_{f^2Z}(F)(f^2X,fY) = -\nabla_{f^2X}(F)(fX,f^2Z) + \nabla_{fZ}(F)(fX,fY), \tag{2.12}$$

Adding (2.10), (2.11) (and 2.12) we get

$$\begin{aligned}2\{\nabla_{f^2X}(F)(f^2Y,fZ) + \nabla_{f^2Y}(F)(f^2Z,fX) + \nabla_{f^2Z}(F)(f^2X,fY)\} \\ = dF(fX,fY,fZ)\end{aligned} \tag{2.13}$$

But for f-almost Kählerian manifold $dF(fX,fY,fZ) = 0$. Thus using this in (2.13) we get

$$\nabla_{f^2X}(F)(f^2X,fZ) + \nabla_{f^2Y}(F)(f^2Z,fX) + \nabla_{f^2Z}(F)(f^2X,fY) = 0$$

Theorem 2.6. A f-Quasi Kählerian manifold is f-Kählerian whenever f is connection preserving.

Proof. Since f is connection preserving, we have

$$\nabla_{fX}fY = \nabla_X Y \tag{2.14}$$

Let the manifold be f-Quasi Kählerian, then

$$\begin{aligned}\nabla_{fX}f(fY) &= -\nabla_{f^2X}f(f^2Y) \\ &= -\{\nabla_{f^2X}f(f^2Y) - f\,\nabla_{f^2X}f^2Y\} \\ &= -\{\nabla_{fX}f^2Y - f\,\nabla_{fX}fY\}\end{aligned}$$

by virtue of (2.14)

$$= -\nabla_{fX}(f)(fY) \quad \text{or} \quad 2\,\nabla_{fX}f(fY) = 0$$

Hence the f-Quasi Kählerian manifold in which f is connection preserving, is f-Kählerian.

THEOREM 2.7. In a f-Kählerian manifold the *nasc* that f preserves connection is that

$$\nabla_{fX} f^2 Y = f \nabla_X Y$$

PROOF. Since the manifold is f-Kählerian, so that

$$\nabla_{fX}(f^2 Y) - f \nabla_{fX} f Y = 0 \tag{2.15a}$$

Let f is connection preserving, then

$$\nabla_{fX} f Y = \nabla_X Y \tag{2.15b}$$

Thus from (2.15)a and (2.15)b we get

$$\nabla_{fX} f^2 Y = f \nabla_X Y$$

Conversely, let

$$\nabla_{fX} f^2 Y = f \nabla_X Y \tag{2.15}$$

from (2.15)a and (2.15) we get $\nabla_{fX} f Y = \nabla_X Y$, that is f is connection preserving.

3. In this section we obtain some results for f-structure manifold with the help of operators $G(X, Y, Z)$ and $J(X, Y, Z)$ defined as follows:

$$\begin{aligned} &\text{a)} \quad G(X,Y,Z) \overset{\text{def}}{=\!=\!=} \nabla_X(F)(Y,Z) + \nabla_Y(F)(X,Z) \\ &\text{b)} \quad J(X,Y,Z) \overset{\text{def}}{=\!=\!=} \nabla_X(F)(Y,Z) + \nabla_Y(F)(Z,X) + \nabla_Z(F)(X,Y) \end{aligned} \tag{3.1}$$

THEOREM 3.1 For an f-almost Kählerian manifold, we have

$$G(fX, f^2Y, f^2Z) + G(fY, f^2Z, f^2X) + G(fZ, f^2X, f^2Y) = 0 \tag{3.2}$$

PROOF. From (3.1), we get

$$G(fX, f^2Y, f^2Z) = \nabla_{fX}(F)(f^2Y, f^2Z) + \nabla_{f^2Y}(F)(fX, f^2Z)$$

Hence

$$\begin{aligned} &G(fX, f^2Y, f^2Z) + G(fY, f^2Z, f^2X) + G(fZ, f^2X, f^2Y) \\ &= - \{\nabla_{fX}(F)(fY, fZ) + \nabla_{fY}(F)(fZ, fX) + \nabla_{fZ}(F)(fX, fY)\} - \\ &- \{\nabla_{f^2X}(F)(f^2Y, fZ) + \nabla_{f^2Y}(F)(f^2Z, fX) + \nabla_{f^2Z}(F)(f^2X, fY)\} \end{aligned}$$

which by virtue f (2.1)b and theorem 2.5, gives 3.2.

THEOREM 3.2. If an f-structure manifold has the following two properties:

a) it is an f-almost Kählerian manifold,
b) it is an f-nearly Kählerian manifold, (3.3)

then

$$\nabla_{f^2Z}(F)(fX,f^2Y) = 2\,\nabla_{fX}(F)(fY,fZ).$$

PROOF. In view of definition (2.1) b, let us put

$$\text{c)}\quad J(X,Y,Z) = \sum_{X,Y,Z} dF(X,Y,Z), \tag{3.3}$$

where $J(X,Y,Z)$ is given by (3.1).

Therefore

$$J(fX,fY,fZ) + G(fX,fY,fZ) = 2\,\nabla_{fX}(F)(fY,fZ) + \nabla_{fZ}(F)(fX,fY)$$

which for fAK an fNK-manifold gives

$$2\,\nabla_{fX}(F)(fY,fZ) + \nabla_{fZ}(F)(fX,fY) = 0$$

or

$$2\,\nabla_{fX}(F)(f^2Y,f^2Z) = -\,\nabla_{f^2Z}(F)(fX,f^2Y)$$

which by virtue of (1.11) yields

$$2\,\nabla_{fX}(F)(fY,fZ) = \nabla_{f^2Z}(F)(fX,f^2Y).$$

THEOREM 3.2. For an f-nearly Kählerian manifold be have

$$\nabla_{f^2X}(F)(f^2Y,fZ) + \nabla_{f^2Y}(F)(f^2X,fZ) = 0 \tag{3.4}$$

PROOF. The proof of the above theorem is obvious.

Remark. The equation (3.4) gives an alternative definition of f-nearly Kählerian manifold.

THEOREM 3.3. For an f-nearly Kählerian manifold

$$J(fX,f^2Y,f^2Z) + J(fY,f^2X,f^2Z) = \nabla_{f^2X}(F)(f^2Y,fZ) + \nabla_{f^2Y}(F)(f^2Z,fX)$$

PROOF. From (3.1) b we get

$$\begin{aligned} & J(fX,f^2Y,f^2Z) + J(fY,f^2X,f^2Z) = \\ & = -\,\{\nabla_{fX}(F)(fY,fZ) + \nabla_{fY}(F)(fX,fZ\} + \\ & + \nabla_{f^2Y}(F)(f^2Z,fX) + \nabla_{f^2X}(F)(fY,f^2Z) + \\ & + \nabla_{f^2Z}(F)(fY,f^2X) + \nabla_{f^2Z}(F)(fX,f^2Y) \end{aligned} \tag{3.5}$$

using (1.10) and (2.1) c, we get the required relation.

THEOREM 3.4. For an f-nearly Kählerian manifold

$$G(f^2X, f^2Y, fZ) + G(fX, f^2Y, f^2Z) + G(f^2X, fY, f^2Z) = 0$$

PROOF. The proof of the above theorem follows immediately from the equations (3.1), (1.11), (1.10) and (3.4).

Corollary. For an f-structure manifold the following identities hold.

$$\begin{aligned} &\text{a) } J(f^2X,f^2Y,fZ) - J(fY,fX,fZ) = 0, \\ &\text{b) } J(fX, f^2Y, fZ) + J(fY, f^2X, fZ) = 0, \\ &\text{c) } J(f^2X,f^2Y,f^2Z) - J(fY,fX,f^2Z) = 0 \end{aligned} \tag{3.6}$$

PROOF. The proof is obvious.

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