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Using coefficient of variation for estimating parameters of some continuous distributions

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1. INTRODUCTION

The method of moments for estimating parameters does not always give useful results, since it often implies to solve some transcendental equations. Some of these equations usually contain the Gamma function.

The purpose of this report it to provide some examples on estimating parameters of some continuous distributions by the aid of the moments method, via the sample coefficient of variation.

2. THE TECHNIQUE

Given a continuous random variable X whose density depends on two unknown parameters, $f(x;\Theta,K)$ where Θ is a scale parameter, K is a shape (or a power) parameter, suppose that there exists $E(X) \neq 0$ and $Var\ (X)$.

If the theoretical coefficient of variation, namely:

$$CV(X) = \frac{[Var(X)]^{1/2}}{E(X)} \tag{1}$$

depends only on K, that is:

$$CV(X) = \varphi(K) \tag{2}$$

then, (2) may be tabulated for various K in the range of variation of this parameter.

Hence, if we estimate $CV(X)$ by:

$$\widehat{CV(X)} = \frac{s}{x} \tag{3}$$

where $\bar{x} = n^{-1}\,\Sigma x_i$, $s = [(n-1)^{-1}\,\Sigma(x_i - \bar{x})^2]^{1/2}$, $x_1, x_2, \ldots, x_n$ being the sample

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values, it follows that if we find in the table the value of the ratio (3), we may detect the estimation of K -- say $\hat{K}$.

This one, is introduced then in the expression of $E(X)$, that is:

$$E(X) = g(\Theta,\hat{K}) \tag{4}$$

which is then equated to $\bar{x}$.

We shall find therefore:

$$\hat{\Theta} = g^{-1}(\hat{K},\bar{x}) \tag{5}$$

The manner is similar when f depends on more than two parameters.

3. SPECIFIC APPLICATIONS

a) Consider the case of the so-called generalized Gamma variable. (See Stacy, 1962.) [5]

$$X : f(x;b,p,\nu) = \frac{\nu}{b\Gamma(p)} \cdot \left(\frac{x}{b}\right)^{p\nu-1} exp\left[-\left(\frac{x}{b}\right)^{\nu}\right] \tag{6}$$

for $x > 0$ and b, p, $\nu > 0$.

Taking into account that:

$$E(X^j) = b^j \frac{\Gamma\left(p + \frac{j}{\nu}\right)}{\Gamma(p)} \tag{7}$$

(see Firkowicz, 1970).[2]

We obtain immediately:

$$CV(X) = \frac{[\Gamma(p)\Gamma(p+2K) - \Gamma^2(p+K)]^{1/2}}{\Gamma(p+K)} \tag{8}$$

where we have put $\nu^{-1} = K$.

For $p = 1$ (Weibull distribution) there exists extensive tables for the purpose of extimation.

Note that if $p = \nu = 1$, one has an exponential distribution for which $CV(X) = 1$. In this case, if we find $\dot{CV}(X) \doteq 1$, then it may serve rather as an empirical criterion for assessing exponentiality than to estimation. (Indeed, there exists other continuous distributions for which (CV(X) = 1 see, for instance, Tweedie, 1957[6] but when someone draws conclusions based only on empirical criteria, he must possess some previous information on the subject.)

b) If in (6) we take $p\nu$ = const = δ let $\nu \to \infty$, we shall obtain the below density:

$$X : f(x;b,\delta) = \delta b^{-\delta} x^{\delta-1}, 0 < x < b, \delta > 0 \tag{9}$$

which is so-called "power distribution" (see Ciechanowicz, 1969,[1] Firkowicz, 1970[2] or Johnson-Kotz, 1970.[3]

In this case:

$$CV(X) = [\delta(\delta + 2)]^{-1/2} \tag{10}$$

Since:

$$E(X^j) = b^j \frac{\delta}{1+\delta} \tag{11}$$

In Table 1, values are given of $CV(X)$ for $\delta = 0.10$ (0.05) 5. Therefore, b is easily estimated as:

$$\hat{b} = \left(1 + \frac{1}{\hat{\delta}}\right)\bar{x} \tag{12}$$

Note that another estimator for b may be the largest order statistic of a given sample, but in some experiments concerning failure analysis of the components, if the life-test in truncated at a certain moment, the largest order statistic is not a realistic estimate.

In this case — say we have the first m failure times — $x_{(1)} < x_{(2)} < \ldots < x_{(m)}$, $m < n$, n being the number of elements put on test, the stimate of b is:

$$\hat{b} = x_{(m)} \cdot \left(\frac{m}{n}\right)^{-\hat{\delta}_{ML}} \tag{13}$$

(see Firkowicz, 1970) [2] where $\hat{\delta}_{ML}$ is the maximum likelihood estimator of δ.

Example: A number of 51 electrical components have been put on a life-test for a certain period of time.

In the table below are recorded the number of failed elements corresponding to six class intervals of time.

Consulting a Firkowic's nomogram (1970, p. 72)[1] it is suspected that power distribution is appropriate.

In this case, we estimate $E(X)$ by:

$$\hat{E}(X) = \sum_{i=1}^{M} \frac{x^{(i)} + x^{(i+1)}}{2} \cdot \hat{p}_i \tag{14}$$

where M is the number of class intervals.

Table 1 Values of CV(X) — power distribution for $\delta = 0.10(0.05)5$.

δ	CV(X)	δ	CV(X)	δ	CV(X)	δ	CV(X)
0.10	2.1822	1.35	0.4704	2.60	0.2887	3.85	0.2108
0.15	1.7623	1.40	0.4583	2.65	0.2851	3.90	0.2085
0.20	1.5076	1.45	0.4472	2.70	0.2828	3.95	0.2063
0.25	1.3328	1.50	0.4364	2.75	0.2763	4.00	0.2041
0.30	1.2038	1.55	0.4264	2.80	0.2732	4.05	0.2020
0.35	1.0976	1.60	0.4167	2.85	0.2692	4.10	0.2000
0.40	1.0206	1.65	0.4076	2.90	0.2654	4.15	0.1980
0.45	0.9535	1.70	0.3872	2.95	0.2617	4.20	0.1961
0.50	0.8884	1.75	0.3904	3.00	0.2582	4.25	0.1939
0.55	0.8452	1.80	0.3824	3.05	0.2548	4.30	0.1921
0.60	0.8006	1.85	0.3748	3.10	0.2516	4.35	0.1903
0.65	0.7625	1.90	0.3674	3.15	0.2484	4.40	0.1883
0.70	0.7274	1.95	0.3604	3.20	0.2454	4.45	0.1867
0.75	0.6967	2.00	0.3536	3.25	0.2418	4.50	0.1850
0.80	0.6682	2.05	0.3471	3.30	0.2390	4.55	0.1832
0.85	0.6428	2.10	0.3408	3.35	0.2364	4.60	0.1814
0.90	0.6189	2.15	0.3348	3.40	0.2331	4.65	0.1798
0.95	0.5976	2.20	0.3289	3.45	0.2306	4.70	0.1782
1.00	0.5773	2.25	0.3234	3.50	0.2276	4.75	0.1765
1.05	0.5590	2.30	0.3180	3.55	0.2253	4.80	0.1751
1.10	0.5415	2.35	0.3131	3.60	0.2245	4.85	0.1735
1.15	0.5256	2.40	0.3072	3.65	0.2203	4.90	0.1720
1.20	0.5103	2.45	0.3029	3.70	0.2177	4.95	0.1705
1.25	0.4963	2.50	0.2975	3.75	0.2152	5.00	0.1690
1.30	0.4828	2.55	0.2936	3.80	0.2132		

Table 2 An example for estimating parameters of the power distribution.

Limits for failure times $[x^{(i)}, x^{(i+1)})$	Number of failed components (m_i)	Frequency $\hat{p}_i = \frac{m_i}{n}$
[0, 50)	20	0.39
[50, 100)	12	0.24
[100, 150)	8	0.15
[150, 200)	6	0.16
[200, 250)	4	0.08
[250, 300)	1	0.02

We obtain $\hat{E}(X) \doteq 100$ hours $[var(X)]^{1/2} \doteq 87$ hours. Hence $\widehat{V}(X) \doteq 0.87$. From table (1) we extract $\hat{\delta} \doteq 0.5$.

Therefore:

$$\hat{b} = 100 \cdot (1 + 2)\ 300 \tag{15}$$

Hence, the distribution function is:

$$X : F(x) = \left(\frac{x}{300}\right)^{1/2}, \quad 0 < x < 300 \tag{16}$$

which agrees with analysis given in Vodă (1974),[7] where a x^2-test (Kozlov and Ushakov, 1970 [4] is applied to fit a power distribution.

c) Consider now the distribution function of the inverse of the power variate.

Table 3 Values of CV(X) — Pareto distribution — for a = 3(1)50.

a	C(X)	a	CV(X)
3	0.5773	27	0.0385
4	0.3536	28	0.0371
5	0.2582	29	0.0357
6	0.2041	30	0.0345
7	01690	31	0.0334
8	0.1443	32	0.0323
9	0.1259	33	0.0313
10	0.1118	34	0.0303
11	0.1005	35	0.0294
12	0.0913	36	0.0286
13	0.0836	37	0.0278
14	0.0772	38	0.0270
15	0.0716	39	0.0262
16	0.0668	40	0.0256
17	0.0626	41	0.0250
18	0.0589	42	0.0244
19	0.0556	43	0.0238
20	0.0527	44	0.0233
21	0.0501	45	0.0227
22	0.0477	46	0.0222
23	0.0455	47	0.0217
24	0.0435	48	0.02128
25	0.0417	49	0.0208
26	0.0400	50	0.0204

A straightforward calculation easily provide:

$$X : F(x;\theta,a) = 1 - \left(\frac{\theta}{x}\right)^a, \quad a, \theta > 0. \, x \geqslant 0 \tag{17}$$

(see also Johnson-Kotz, 1970, p. 234).

The above variate is just a Pareto one. We have therefore:

$$E(X) = \theta a(a-1)^{-1} \text{ if } a > 1 \tag{18}$$

$$Var(X) = \theta^2 a(a-1)^{-2}(a-2) \text{ if } a > 2 \tag{19}$$

from which we get:

$$CV(X) = [a(a-2)]^{-1/2} \text{ for } a > 2 \tag{20}$$

I table 3 values are given of V(X) for a = 3(1)50.

REFERENCES

1. Ciechanowicz, K.: Generalized gamma distribution and poker distribution as failure distributions of the Components. Doctoral thesis: Institute of automation. Polish Academy of Sciences. Warsaw (in Polish.) (1963.)
2. Firkowicz, Sz.: Statistical inference from samples W.N.T., Warsaw (in Polish). (1970.)
3. Johnson, N. L. and Kotz, S.: Distributions in Statistics Continuous Univariate distributions. Vol. 1, Houghton Mifflin Co., Boston (1970).
4. Kozlov, B. A., y Ushakov, I. A.: Reliability handbook. Holt, Rinehart and Winston Inc., New York (1970).
5. Stacy, E. W.: A Generalization of the Gamma Distribution. Am. Math. Statist., p. 1187-1192 (1962).
6. Tweedie, M. C. K.: Statistical properties of the inverse Gaussian Distribution I. Am. Math. Statist., *28*, pp. 362-377 (1957).
7. Vodă, V. Gh.: On a Statistical Distribution Used in Reliability Theory and Life-testing of the Electrical Components. (Paper presented at Fourth Iranian Conference on Electrical Engineering; May 12-16, 1974, Shiraz, Iran. To appear in Iranian Journal of Science and Technology.) (1974).