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INDICATORS AND INCOMPLETENESS OF PEANO ARITHMETIC*

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ABSTRACT

In this article we present the Method of Kirby, París (1976) for proving the existence of arithmetical sentences, which are neither provable nor refutable in Peano Arithmetic of natural numbers.

INDICADORES E INCOMPLETITUD DE LA ARITMETICA DE PEANO

RESUMEN

En este artículo presentamos el método de Kirby y París (1976) para demostrar la existencia de sentencias aritméticas no probables ni refutables en la aritmética de Peano de los números naturales.

Sec. O. Basic notions. We use standard logical and set theoretical notation. Peano Arithmetic (short-*PA*) is a first order theory of natural numbers formulated in a formal language L_{PA} which contains: two binary operation symbols $+, \cdot$ for addition and product and constants $\underline{0}, \underline{1}$ denoting the numbers 0 and 1, respectively. For each $n \in \omega$ (where ω is the set of natural numbers) one defines inductively the constant term $\underline{n}$ denoting the number n. The axioms of *PA* are the finite number of formulas expressing the basic properties of $+, \cdot$, and constants

1. $x + \underline{1} \neq \underline{0}$
2. $x + \underline{1} = y + \underline{1} \rightarrow x = y$
3. $x + \underline{0} = x$
4. $x + (y + \underline{1}) = (x + y) + \underline{1}$
5. $x \cdot \underline{0} = \underline{0}$
6. $x(y + \underline{1}) = xy + x$

and the infinite induction scheme

$$\text{(Ind)} \quad \phi(\underline{0}) \;\&\; \forall x[\phi(x) \rightarrow \phi(x+\underline{1})] \rightarrow \forall x \phi(x),$$

for an arbitrary formula ϕ of *PA*.

The usual ordering $\leqslant$ is defined by the formula $x \leqslant y \; \exists z[x + z = y]$ and $x < y$ is equivalent then to

$$\exists z[x + (z + \underline{1}) = y].$$

The minimum principle is the scheme

$$\text{(Min)} \quad \exists x\, \phi(x) \rightarrow \exists x\, [\phi(x) \;\&\; \forall y < x \;\neg\; \phi(y)]$$

(Each nonempty, definable set has a smallest element).

The maximum principle is the scheme

$$\text{(Max)} \quad \exists x\, \phi(x) \;\&\; \exists z\, \forall x\, [\phi(x) \rightarrow x \leqslant z] \rightarrow$$
$$\rightarrow \exists z\, \{\phi(z) \;\&\; \forall x\, [\phi(x) \rightarrow x \leqslant z]\}$$

(Each nonempty, definable and bounded set has a greatest element). Both (Min) and (Max) schemes are equivalent to (Ind) this is proved using only the remaining axioms 1.-6.

The standard model of *PA*, denoted ω, is the structure $< \omega, +, \cdot, 0, 1 >$, where $+$ and $\cdot$ are the ordinary operations of addition and product. Since we do not distinguish between isomorphic structures, a nonstandard model is any, which is not isomorphic to the standard one. Existence of nonstandard models of *PA* follows immediately from the Löwenheim-Skolem theorem: *PA* as a consistent first order theory has models in every uncountable cardinality. Using the Compactness theorem one proves easily, that there are countable nonstandard models of *PA*.

Each model M of *PA* contains (up to isomorphism) the standard one ω. This follows from the fact, that all sentences of the form $\underline{n} + \underline{m} = \underline{n+m}$ and $\underline{n} \cdot \underline{m} = \underline{n \cdot m}$ are provable in *PA*. Since the sentences $x \leqslant \underline{n} \rightarrow x = \underline{0} \vee x = \underline{1} \vee \ldots \vee x = \underline{n}$ are also provable, we see, that ω is in fact an initial part of M (with respect to the ordering $\leqslant$ of M). It follows also, that M is nonstandard if and only if it contains elements c with the property that $M \models \underline{n} < c$ holds for all $n \in \omega$. All such elements are called nonstandard or infinite.

If M is nonstandard and countable, it is easy to determine the order type of $\leqslant$ of M. Since the standard members form an initial part of M we look only at the infinite elements of M. We claim, that they have the order type that of $\mathbb{Z} \times \mathbb{Q}$ ($\mathbb{Z}$ are the integers and $\mathbb{Q}$ the rationals) ordered antilexicographically.

To see this, let $c > \omega$ (i.e. c is infinite) and put $[c] = \{c + n : n \in \omega\} \cup \{c - n : n \in \omega\}$ (note, that since $c > n$ in M for all $n \in \omega$, the element $c - n$ is well defined). Thus, each class $[c]$ is order isomorphic to $\mathbb{Z}$. Now, the induced or-

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dering of $[c]$'s is that of $\mathbb{Q}$. Indeed, $[c]<[c+c]$ hence, there is no greatest element (if A, B are subsets, we write $A<B$ if $x<y$ for all $x \in A$ and $y \in B$). Since $[c]=[c+1]$ and one of c, $c+1$ is even, the class $\left[\frac{1}{2}c\right]$ is well defined and we see that $\left[\frac{1}{2}c\right]<[c]$ thus, there is no smallest class. Finally, if $[c]<[d]$, then we have $[c]<\left[\frac{1}{2}(c+d)\right]<[d]$, thus the ordering of $[c]$'s is dense. By the well known Cantors's theorem it is isomorphic to $\mathbb{Q}$ and our claim is proved.

We are particulary interested in countable nonstandard models M and their initial segments.

We say, that $I \subseteq M$ is an initial segment of M and write $I \subseteq_e M$ if $x \leqslant y \in I$ implies $x \in I$ (i.e. I is an initial segment in usual sense) and, in addition, I is closed under succesor: $x \in I$ implies $x+1 \in I$. For example we have $\omega \subseteq_e M$, for arbitrary model M. If $c \in M$ and $c>\omega$ then

$$I=\{a \in M: \exists n \in \omega\, [a<c+n]\} \subseteq_e M.$$

This segment is not a model, not even a substructure of M: indeed, $c \in I$ but $c \in I$ but $c+c \notin I$.

We shall apply the usual coding technique. The formula Seq (n) describes the sequence numbers

$$n=p_0^{x_0+1} \cdot p_1^{x_1+1} \cdots p_m^{x_m+1},$$

where p_n is the increasing enumeration of prime numbers. The sequence number n is a code of the finite sequence

$$<x_0, x_1, \ldots, x_m>.$$

We have also the arithmetical functions $\ell h(n)=m+1$ (the length of the sequence) and $(n)_i=x_i$ (the i-th term of the sequence).

The pairing function $j(x, y)=\frac{1}{2}[(x+y+1)(x+y)]+x$ enumerates all the ordered pairs. We put $j_2=j$ and inductively $j_{n+1}(x_1 \ldots x_{n+1})=j(j_n(x_1 \ldots x_n), x_{n+1})$ and j_{n+1} enumerates then all ordered n-tuples. The functions $k_1 \ldots k_n$ are converses to j_n i.e. $j_n(k_1(x), \ldots, k_n(x))=x$, for all x.

If f is a function with finite domain then the code of f is $c_f=\prod_{x \in \mathrm{Dom}(f)} p^{f(x)+1}$. Note, that if M is a model, f is definable in M and Dom(f) is bounded then c_f is defined and is in M.

A formula ϕ is bounded if it has no quantifiers or if all the quantifiers in ϕ are bounded i.e. are of the form

$$\exists x\, [x \leqslant y \,\&\ldots] \text{ or } \forall x\, [x \leqslant y \rightarrow \ldots].$$

Σ_1-formulas have the form $\exists x_1 \ldots x_n$ with ϕ bounded Π_1-formulas have the form $\forall x_1 \ldots x_n\, \phi$, ϕ-bounded. Also, a formula which is equivalent (in PA) to a Σ_1-or a Π_1-formula is said to be Σ_1 or Π_1, respectively. Δ_1-formulas are those, which are both Σ_1 and Π_1. For example all recursive functions are Δ_1-definable. If ϕ is Δ_1 then ϕ is absolute i.e. if $M_1 \subseteq M_2$ are models, then

$$M_1 \models \phi\, [a_1 \ldots a_n] \text{ iff } M_2 \models \phi\, [a_1 \ldots a_n],$$

for all $a_1 \ldots a_n$ from M_1.

If $\phi(x_1 \ldots x_n, y)$ is a formula with the free variables indicated and such that $PA \vdash \forall x_1 \ldots x_n \exists !\, y\, \phi$ holds, then for any model M the set

$$\phi^M=\{<a_1 \ldots a_n, b>: M \models \phi\, [a_1 \ldots a_n, b]\}$$

is a función: $\underbrace{M \times \ldots \times M}_{n\text{- times}} \rightarrow M$. If, in addition, ϕ is Σ_1, then it is also Π_1 and hence absolute, thus we have

$$\phi^{M_1}(a_1 \ldots a_n)=\phi^{M_2}(a_1 \ldots a_n),$$

for models $M_1 \subseteq M_2$ and arbitrary $a_1 \ldots a_n \in M_1$.

We shall use the following theorem called Dirichlet's Principle: If $f: M \rightarrow M$ is definable over M and there is a definable subset $X \subseteq M$ such that X is unbounded and $f(x) \leqslant a$, for $x \in X$, then there is a definable, unbounded subset $Z \subseteq X$ and an element $b \leqslant a$ such that $f(x)=b$, for $x \in Z$.

Sec. 1. Indicators. In this section we introduce the notion of indicator and prove the main theorem on incompleteness of PA. The existence of indicators will be shown in next sections.

Definition. A PA-formula $Y(x, y, z)$ is an indicator (for models of PA) if Y is Σ_1 and:

i) $PA \vdash \forall x, y\, \exists! z\, Y(x, y, z)$ holds.

ii) For each countable, nonstandard model M of PA and all $a, b \in M$ we have

$$Y^M(a, b)>\omega \text{ iff } \exists I \subseteq_e M\, [I \models PA \,\&\, a \in I<b]$$

Remark. By (i), the set

$$Y^M=\{<a, b, c>: M \models Y\, [a, b, c]\}$$

is a function from $M \times M$ to M and hence we use more convenient notation $Y(x, y)=z$ instead of $Y(x, y, z)$ holds. As remarked in Sec. 0. a total Σ_1-defined function is in fact Δ_1 and hence absolute.

The term "indicator" is justified by ii): given $a, b \in M$, then Y indicates (by taking an infinite value) if there in an initial segment $I \subseteq_e M$, which is itself a model of PA and such that $a \in I<b$.

Throughout the rest of this section we assume that indicators exist.

Theorem 1.1. If Y is an indicator, then the sentence Θ_Y $\forall x, z \, \exists y \, [Y(x,y) > z]$ is independent (i.e. neither provable nor refutable) in *PA*.

Proof. We shall prove, that both Θ_Y and $\neg \Theta_Y$ are consistent with *PA*. We make use of the classical MacDowell, Specker theorem (1959): "Each model M of *PA* has a proper, elementary extension M_1 of the same cardinality such that $M \subseteq_e M_1$." (In Sec. 5. we prove a stronger form of this theorem in the case of countable models). Let M be countable and nonstandard. Suposse, that from some $n \in \omega$ and some $a \in M$ we have $M \models \forall y \, [Y(a, y) \leqslant \underline{n}]$. Let M_1 be such as in McDowell, Specker theorem. Then we have (since $M < M_1$)

$$M_1 \models \forall y \, [Y(a, y) \leqslant \underline{n}]$$

On the other hand, if $b \in M_1 \setminus M$, then $a \in M < b$ and since $M \subseteq_e M_1$ we infer from (ii), that $Y^{M_1}(a, b) > \omega$ i.e.

$M_1 \models Y(a, b) > \underline{m}$, for all $m \in \omega$, which contradicts our assumption. Thus, we have proved, that

$$M \models \forall x \, \exists y \, [Y(x, y) > \underline{n}]$$

holds for all $n \in \omega$. Among all M's there are such, which are elementarily equivalent to ω. Thus, we have in particular $\omega \models \forall x \, \exists y \, [Y(x, y) > \underline{n}]$, for all $n \in \omega$, which implies immediately $\omega \models \Theta_Y$. To see, that $\neg \Theta_Y$ is consistent, we prove first the following.

Lemma 1.2. (Friedman). Each countable, nonstandard model M contains a proper, nonstandard initial segment $M_1 \models PA$. Proof of the Lemma 1.2.: Choose $a, b \in M$, such that $a \in \omega < b$. Then $Y^M(a, b) > \omega$. Consider the set

$$X = \{x \in M \colon Y^M(a, x) \leqslant Y^M(x, b) \;\&\; a \leqslant x < b\}.$$

We see that

1. $X \neq \phi$. Indeed, $Y(a, a) < \omega$ and $Y(a, b) > \omega$ hence $a \in X$.

2. X is parametrically definable in M. Indeed, Y is in fact a *PA*-formula.

3. X is bounded: by definition $X < b \in M$.

Thus, by the Maximum Principle in M, X has the greatest element:

$$d = \max X$$

We claim: $Y^M(a, d) > \omega$. Indeed, from $Y^M(a, d) \in \omega$, follows $d \in \omega$, thus $d + 1 \in \omega$ and consequently

$$Y^M(a, d + 1) \in \omega.$$

Since $d + 1 \in \omega < b$, we have $Y^M(d + 1, b) > \omega$ i. e. $Y^M(a, d + 1) < Y^M(d + 1, b)$. Further, $a < d + 1$, for $a \leqslant d$ and obviously $d + 1 < b$, since $Y^M(d + 1, b) > \omega$. We showed, that from $Y^M(a, d) < \omega$ follows $d + 1 \in X$, a contradiction. From the claim follows:

$$\omega < Y^M(a, d) \leqslant Y^M(d, b)$$

Thus, $d > \omega$ and hence the segment M_1 s. t. $d \in M_1 < b$ is proper and nonstandard, which finishes the proof of the Lemma.

Remark. The only requirement on b was $\omega < b \in M$ and the segment M_1 satisfies $M_1 < b$. Thus, we see that each nonstandard countable $M \models PA$ contains arbitrarily "short" nonstandard segments $M_1 \models PA$.

Now, let us introduce the notion of index.

Definition. The index of an initial segment I of a countable nonstandard model M (with respect to a given indicator Y) is the set

$$\mathrm{ind}_M(I) = \{z \in M \colon \forall a \in I \, \exists b \in I \, [Y^M(a, b) > z]\}.$$

Consider the case when I is itself a model of *PA*. Then $Y^I \colon I \times I \to I$ and moreover $Y^I = Y^M \cap I^2$, by absoluteness. It follows, that $\mathrm{ind}_M(I) \subseteq I$ (and also $z' < z \in \mathrm{ind}_M(I)$ implies $z' \in \mathrm{ind}_M(I)$). On the other hand, since Y^M can be replaced by Y^I, we see immediately that in this case

$$\mathrm{ind}_M(I) = I \text{ iff } I \models \Theta_Y$$

We shall prove

Lemma 1.3. Each countable, nonstandard model M contains an initial segment $M_1 \models PA$ with $\mathrm{ind}_M(M_1) < M_1$.

Thus, $M_1 \models \neg \Theta_Y$, which, by the above given remark gives the desired independency of Θ_Y from *PA*.

Proof of Lemma 1.3.: by Friedmann's Lemma, there exist $a, b \in M$ such that

$$\omega < a < b \in M \text{ and } Y^M(a, b) > \omega.$$

We find and element c with the property

$$\omega < \max \{Y^M(a, y) : y \leqslant c\} < a$$

If $Y^M(a, y)$ is always $< a$, then take $c = b$. Otherwise, let $d = \inf \{y \colon Y^M(a, y) \geqslant a\}$. We have

$$\max \{Y^M(a, y) : y \leqslant d - 1\} < a.$$

On the other hand, since $Y^M(a, d) \geqslant a > \omega$, there is a segment $a \in I < d$. Obviously, then $a \in I < d - 1$, thus

$$Y_M(a, d - 1) > \omega.$$

Hence $c = d - 1$ has the required property. Now, let $e \leqslant c$ be such that

$$Y^M(a, e) = \max \{Y^M(a, y) : y \leqslant c\}$$

Thus, we have

$$\omega < Y^M(a, e) < a$$

Let M_1 be an initial segment of M such that

$$M_1 \models PA \text{ and } a \in M_1 < e$$

Then, we have $z \in \text{ind}_M(M_1)$ implies

$$\forall u \in M_1 \ \exists v \in M_1 \ [Y^M(u, v) > z].$$

Substituting $u = a$ we obtain $\exists v \in M_1 \ [Y^M(a, v) > z]$, which implies $\exists v \leqslant c \ [Y^M(a, v) > z]$, since

$$M_1 < e \leqslant c.$$

Hence $z < Y^M(a, e) < a$.

Thus, $\text{ind}_M(M_1) < a \in M_1$, which finishes the proof of the Lemma 1.3. and of the Theorem 1.1.

Sec. 2. Finite Games. In this section we recall some notions of Game Theory, which will be used in the construction of indicators. Let $c \in \omega$ be a fixed number and let S be a set of finite sequences of the length $\leqslant 2c$. We say, that S is a set of positions if the following two conditions are satisfied

1. if $s = \langle x_0, \ldots, x_n \rangle$ is in S, so is each restriction $s \restriction j = \langle x_0, \ldots, x_{j-1} \rangle$ of s.

2. if $s \in S$ and $\ell h(s) < 2c$, then there is an $s' \in S$ such that s' extends s properly (i.e. $\ell h(s') > \ell h(s)$)

It follows, that if $S_j = \{s \in S : \ell h(s) = j\}$, then

$$S = \bigcup \{S_j : 0 \leqslant j \leqslant 2c\}$$

and each S_j is nonempty. An arbitrary subset $A \leqslant S_{2c}$ determines a game G_A, which is played by two players (denoted I and II) as follows: player I begins by choosing an s_1 from S_1, then player II chooses an extension s_2 of s_1 from S_2, then again player I chooses an s_3 from S_3 extending s_2 and so on. Each player makes c steps, thus, at the end of G_A, a sequence $s_{2c} \in S_{2c}$ is fixed. (The terms of s_{2c} with even indices have been choosen by player I and those with odd indices by player II, respectively). If $s_{2c} \in A$, then player I wins G_A, otherwise II wins.

A strategy (for player I) is an arbitrary function

$$\sigma : \bigcup_{j<c} S_{2j} \to S,$$

such that $\sigma(s)$ extends s by one term (i. e. $\sigma(s) \in S_{2j} + 1$ if $s \in S_{2j}$). A position s is consistent with σ if s has the form $\langle \sigma(\phi), x_1, \sigma(x_1), \ldots, x_n, \sigma(x_1 \ldots x_n), \ldots \rangle$ i. e. if $s \restriction 2_{j-1}$). A given strategy σ is a winning strategy (for player I) if each position $s \in S_{2c}$, consistent with σ belongs to A. Intuitively, the player I always wins if he only uses the strategy σ in the course of the game G_A. Analogously, the notion of a winning strategy is introduced for player II. A given game G_A is said to be determined if one of the players, I or II, has a winning strategy. One proves (using finite induction, only) that each game G_A of the form described above (that is: of finite lenght) is determined.

We are particularly interested in such games, which can be defined in PA. The sets S and A are replaced then by defining PA-formulas ϕ_s and ϕ_A, respectively.

Let $E(\phi_S, \phi_A)$ denote the conjunction of the following formulas

1. $\exists s \, \phi_S(s)$

2. $\forall s \, [\phi_S(s) \to \text{Seg}(s) \,\&\, \ell h(s) \leqslant 2c]$

3. $\forall s \, [\phi_S(s) \to \forall j < \ell h(s) \, \exists s' \, \forall i < \ell h(s') \, [(s')_i = (s)_i \,\&\, \phi_S(s')]]$

4. $\forall s \, [\phi_S(s) \,\&\, \ell h(s) < 2c \to \exists s' \, [\phi_S(s') \,\&\, \ell h(s') > \ell h(s) \,\&\, \forall j < \ell h(s) \, [(s)_j = (s')_j]]]$.

5. $\forall s \, [\phi_A(s) \to \phi_S(s) \,\&\, \ell h(s) = 2c]$

Thus, the formula $E(\phi_S, \phi_A)$ says that ϕ_S defines a set of positions of length $\leqslant 2c$ and ϕ_A defines a subset of positions of length $= 2c$. If, in addition ϕ_S is limited (i. e. $\exists b' \, \forall s [\phi_S(s) \to s \leqslant b']$), then the strategies, as functions on finite domain, can be encoded by single members and hence the determinacy of G_A can be formalised and proved in PA. We easily construct formulas $W_I(\sigma)$ and $W_{II}(\sigma)$ saying that σ is (a code of a) winning strategy for the player I and the II, respectively.

Then we have

$$PA \vdash E(\phi_s, \phi_A) \,\&\, \exists b' \, \forall s \, [\phi_s(s) \to s \leqslant b'] \to$$
$$\to \exists \sigma \, W_I(\sigma) \vee \exists \sigma \, W_{II}(\sigma)$$

Note that the formulas W_I and W_{II} contain (besides the variable b running through strategies) a parameter c (determining the number of steps of each player in G_A). There are possibly other parameters for instance to make the formula ϕ_s limited.

Sec. 3. Strong Segments. Let M be a countable nonstandard model of PA. We say, that an initial segment $I \subseteq M$

is strong (in M) if the following holds: for each function $f: M \to M$, definable over M, there is an element $e, I < e \in M$, such that

(1) for all $x \in I$, either $f(x) \in I$ or else $f(x) > e$.

We shall give two simple lemmas which will be useful later. Let $b^< = \{x \in M : x < b\}$ and let Fnc (f, b) be an arithmetical formula, which stands for f is (a code of) a function from $b^<$ into $b^<$ i. e.

$$\forall x, y \leq f\, [p_x^{y+1} \mid f \;\&\; p_x^{y+2} \nmid f \to x < b \;\&\; y < b] \;\&$$

$$\&\; \forall x < b\; \exists y < b\; [p_x^{y+1} \mid f]$$

Clearly, Fnc is Δ_1 and hence absolute.

Lemma 3.1. If I is strong in M, then for all $b, I < b \in M$, and all $f \in M$ s.t. $M \models \mathrm{Fnc}\,[f, b]$ there is an $e, I < e \in M$ such that (1) holds.

Proof. Let b and f be as stated above. Put

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{Dm}(f) \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $g: M \to M$ is definable over M and since I is strong in M we find an $e, I < e \in M$, such that (1) holds for g. Thus, (1) holds also for f since f is a restriction of g.

Lemma 3.2. Let I be an initial segment of M. If for some $b, I < b \in M$, and all $f \in M$, with $M \models \mathrm{Fnc}\,[f, b]$, there is an $e, I < e \in M$, such that (1) holds, then I is strong in M.

Proof. Let $g: M \to M$ be definable over M. Define $f: b^< \to b^<$ as follows

$$f(x) = \begin{cases} g(x), & \text{if } g(x) < b \\ b - 1, & \text{otherwise} \end{cases}$$

Now, $f \in M$, for g is definable and also $M \models \mathrm{Fnc}\,[f, b]$. By assumption, there is an $e', I < e' \in M$, c.f. (1) holds for f. Let $e = \min\{e', b-1\}$. Then, for $x \in I$ (in particular $x < b$) if $g(x) \notin I$, then either $g(x) = f(x)$ and hence $g(x) > e$ or else $g(x) > b - 1$ and hence $g(x) > e$. Thus, (1) holds for g and e and the Lemma is proved.

Corollary 3.3. (Absoluteness of strong segments). Let M_1, M_2 be two nonstandard models and $M_1 \subseteq_e M_2$ be an initial segment of M_2. An initial segment $I \subseteq_e M_1$ of M_1 is strong in M_1 if and only if it is strong in M_2.

Proof. Assume I is strong in M_2. Choose a $b, I < b \in M_1$ and let $f \in M_1$ be such that $\models \mathrm{Fnc}\,[f, b]$. By absoluteness of Fnc we have $M_2 \models \mathrm{Fnc}\,[f, b]$ and by Lemma 3.1., there is an $e, I < e \in M_2$ (in fact $e \in b < M_1$) such that (1) holds. Thus, by Lemma 3.2., I is strong in M_1.

Conversely, assume now, that I is strong in M_1. Take a $b, I < b \in M_1$, and let $f \in M_2$ be such that $M_2 \models \mathrm{Fnc}\,[f, b]$. Since $f \leq b\, p_b^b \in M_1$, f is already in M_1 and by absoluteness of Fnc we have $M_1 \models \mathrm{Fnc}\,[f, b]$. Now, since I is strong in M_1, σ_1 Lemma 3.1. there is an $e, I < e < M_1 \subseteq_e M_2$ with the property (1). From Lemma 3.2., we infer that I is strong in M_2.

Now, we generalize slightly the notion of indicator. For each countable, nonstandard model M, let Q^M be a family of some initial segments of M. Let Q be the union of all such Q^M's. We say, that a Σ_1-formula Y is a Q-indicator if the condition i) of the Definition of Sec. 1. holds and a modified version of ii):

ii') for each countable, nonstandard model M and $a, b \in M$ we have

$$Y^M(a, b) > \omega \text{ iff } \exists I\, [I \in Q^M \;\&\; a \in I < b].$$

Thus, if we put $Q_m^M = \{I : I \subseteq_e M \;\&\; I \models PA\}$, then a Q_m-indicator is an indicator for models of PA in the sense of the definition of Sec. 1.

We put $Q_s^M = \{I : I \subseteq_e M \;\&\; I \text{ is strong in } M\}$. We shall prove now the existence of Q_s-indicators (i.e. indicators for strong segmentes).

Theorem 3.4. There exists a Q_s-indicator.

Proof. First, we describe a certain arithmetical game. The corresponding formulas W_I, W_{II} (see Sec. 2.) contain a free variable σ (denoting a strategy) and parameters a, b, c. Hence, the game under consideration will be denoted by $G_c(a, b)$.

The positions of $G_c(a, b)$ are sequences $s = \langle x_0, \ldots, x_n \rangle$, with $n \leq 2c - 1$, and such that.
The terms x_{2j} (i.e. the steps of player I) are either arbitrary numbers $x \leq b$ or else numbers $f > b$ satisfying $\mathrm{Fnc}(f, b)$.

The terms x_{2j+1} (i.e. the steps of player II) are numbers 0 or 1 if x_{2j} was $\leq b$, and an arbitrary $e \leq b$ if x_{2j} was an $f > b$.

The intuitive idea lying behind the game $G_c(a, b)$ can be described thus: assume $a < b$ and imagine, there is a strong segment I satisfying $a \in I < b$. The moves of player I can be interpreted as some questions concerning I and that of player II as answers.

Namely, the move $x \leq b$ of player I is understood as the question "is x in I?" and a move 0 or 1, which follow, as the answer "yes" or "no", respectively. A move e of player II following a move f of player I is interpreted as an example of a number e satisfying the condition (1) (see Lemma 3.1.). Player I wins $G_c(a, b)$ if player II, in the course of the game, answers inconsistently with the fact, that I is a strong initial segment, with $a \in I < b$. Hence, the set A of winning positions (for player I) is described thus: A consists of all the positions $s = < x_0, \ldots, x_{2c-1} >$, which satisfy one of $1_A - 6_A$ below:

1_A. for some j, x_{2j} is a and x_{2j+1} is 1 (i.e. "no").

2_A. x_{2j} is b and x_{2j+1} is 0.

3_A. x_{2j} is an $x \leqslant b$ and x_{2j+1} is 0, x_{2i} is an $y \leqslant b$ and x_{2i+1} is 1 and $y = x + 1$.

4_A. x_{2j} is $\leqslant b$ and x_{2j+1} is 0, x_{2i} is $\leqslant b$ and x_{2i+1} is 1 and $x_{2i+1} \leqslant x_{2i}$.

5_A. x_{2j} is an $f > b$ and answer e follows, and some x_{2i} is e and answer 0 (i.e. "yes") follows.

6_A. x_{2j} is an f with answer e, some x_{2i} is an $\leqslant b$ with answer "yes", some x_{2k} is $f(x)$ with answer "no" and still $f(x) \leqslant e$.

The meaning of $1_A - 6_A$ is clear: 1_A means that $a \notin I$ and 2_A that $b \in I$. 3_A and 4_A mean, that I is not an initial segment and finally 5_A and 6_A contradict the conclusion of the Lemma 3.1. and hence means that I is not strong.

The above described conditions yield PA-formulas ϕ_S and ϕ_A, with free variables precisely s, a, b, c, which are easily seen to be Δ_1 and such that $PA \vdash E(\phi_s, \phi_A)$ holds. Moreover, since all moves are bounded from above, the set of positions is provably finite and hence the determinacy of $G_c(a, b)$ is formalizable and provable in PA. Let $W_I(b, a, b, c)$ and $W_{II}(b, a, b, c)$ have the same meaning as in Sec. 2. We observe the following: if c is small, then player II has a winning strategy e.g. if $c = 1$ (1 question and 1 answer,), then player II answers "yes" for a and "no" for b and arbitrary otherwise. None of $1_A - 6_A$ is fulfilled and hence player II wins. It is obvious, that if player II has a winning strategy b in $G_c(a, b)$, he has also one in any $G_{c'}(a, b)$, with $c' < c$ (he plays the same b in fewer number of steps only).

On the other hand if e.g. $c > b$, then player I can win $G_c(a, b)$: he asks all numbers $x \leqslant b$. Indeed, if all the answers are then "yes", so is for the question $x = b$ and 2_A is fulfilled. Otherwise the set of $x's$ with answer "no" is definable and nonempty and hence has a smallest element x_0. If $x_0 = 0$, then either 1_A holds or else 4_A since $0 \leqslant a$. If $x_0 > 0$, then $x_0 - 1$ is followed with "yes" and 3_A holds. It is also obvious, that if player I has a winning strategy σ in a given $G_c(a, b)$, he has also one in any $G_{c'}(a, b)$, with $c' > c$ (he plays the same σ during the first c steps, other he plays arbitrarily).

The above observations can be easily formalized and proved in PA and hence we obtain, that the following relations are true

(S_1) $PA \vdash a < b \rightarrow \exists \sigma W_{II}(\sigma, a, b, 1)$

(S_2) $PA \vdash \exists \sigma W_{II}(\sigma, a, b, c) \& c' < c \rightarrow \exists \sigma W_{II}(\sigma, a, b, c')$

(S_3) $PA \vdash a < b \rightarrow \exists \sigma W_I(\sigma, a, b, b + 1)$

(S_4) $PA \vdash \exists \sigma W_I(\sigma, a, b, c) \& c' > c \rightarrow \exists \sigma W_I(\sigma, a, b, c')$.

Now, we put $Z_{a,b} = \{c \colon \sigma W_{II}(\sigma, a, b, c)\}$. Thus, the set $Z_{a,b}$ is definable, by (S_1) nonempty and by (S_3) and (S_4) bounded. We define our indicator Y by

$$Y(a, b) = \begin{cases} 0, & \text{if } a \geqslant b \\ \max Z_{a,b}, & \text{if } a < b \end{cases}$$

Obviously, we have $PA \vdash \forall a, b \exists ! c[Y(a, b) = c]$. From (S_3) and (S_4) follows, that $Y(a, b) = c$ is equivalent to

$[a \geqslant b \& c = 0] \vee [a < b \& \exists \sigma W_{II}(\sigma, a, b, c) \& \exists \sigma W_I(\sigma, a, b, c + 1)]$

Since, as observed earlier, both ϕ_S and ϕ_A are Δ_1, so are W_I and W_{II} and the above shows, that Y is Σ_1. It remains to prove, that the condition (ii′) holds true.

Let M be a countable and nonstandard model and let $a, b \in M$. Assume $Y^M(a, b) = c > \omega$. This means, that player II has a winning strategy $\sigma \in M$ in $G_c(a, b)$ interpreted now in M, for an infinite c. Since M is countable, so is the number of all possible questions of player I. Ley q_n be an enumeration of all the moves of player I and let

$$s_n = < q_0, \ldots, q_n > * \sigma = < q_0, \sigma(q_0), \ldots, q_n, \sigma(q_0 \ldots q_n) >$$

Thus, s_n is a position in $G_c(a, b)$ consistent with σ, and hence it can be extended to a winning position (for player II). It follows, that each s_n satisfies the negation of each $1_A - 6_A$. Moreover, each possible question of player I occurs as a term in some s_n, for large enough n. To simplify the notation we make the following convention if $x \leqslant b$ or $f > b$ and $M \models \mathrm{Fnc}\,[f, b]$, then both x and f are possible moves of player I. Hence $x = q_n$ and $f = q_m$, for some n and m. The answers of player II for these questions, according to the winning strategy b are the values $\sigma(s_n)$ and $\sigma(s_m)$, respectively. We shall write more directly $\sigma(x)$ and $\sigma(f)$, respectively.

Define $I = \{x \in M \colon \sigma(x) = \text{"yes"}\}$. We show, that I is a strong initial segment of M and $a \in I < b$. Indeed, $\sigma(a) =$ = "yes" and $\sigma(b) =$ "no" according to 1_A and 2_A, respectively. Thus, $a \in I$ and $b \notin$ If, $x \in M$, and $b(x) =$ yes, then from 3_A follows $b(x + 1) =$ yes, thus $x + 1 \in I$. Similarly, if $x \in I$ and $y \leqslant x$, then 4_A implies $\sigma(y) =$ yes, thus $y \in I$. We have shown, that I is an initial segment of M and $a \in I < b$. To see that I is strong we apply the Lemma 3.2. Let $f \in M$ be such, that $M \models \mathrm{Fnc}\,[f, b]$. Then f is an admissible question in $G_c(a, b)$. Let $e = \sigma(f)$. From 5_A follows, that $e \notin I$, hence $I < e$. Let $x \in$ I and suppose $f(x) \notin I$. Thus $\sigma(x) =$ = yes, $\sigma(f(x)) =$ no and from 6_A follows that $f(x) > e$ i.e. the condition (1) holds. By the Lemma 3.2 I is strong in M.

Now, we prove the implication from right to left of the equivalence in (ii'). Let $I \underset{c}{\subseteq} M$ be strong and $a \in I < b$. We have to show, that $Y^M(a, b) > \omega$. It is easy to see, that $\omega \subseteq Z_{a,b}$. Indeed, if $c \in \omega$, then player II uses the following strategy: for $x \leqslant b$, $\sigma(x) =$ yes iff $x \in I$ and for f with $M \models \mathrm{Fnc}\,[f, b]$, $\sigma(f) =$ an e with the property (1). Such an e exists by the Lemma 3.1. Since the positions have the length $\leqslant 2c \in \omega$, thus finite (in real sense not that of M) this strategy σ is explicitly definable in M by a PA-formula and hence $\sigma \in M$. Thus $c \in Z_{a,b}$. Now, a definable set containing all the standard numbers must contain also some nonstandard ones. Indeed, if not, then the complement consists precisely of nonstandard numbers and is definable and hence has the smallest element x_0. But, then $x_0 - 1 \in \omega$ follows, a contradiction.

Thus, there are $c \in M$, $c > \omega$ and $c \in Z_{a,b}$ and consequently $Y^M(a, b) = \max Z_{a,b} > \omega$. This finishes the proof of the Theorem 3.4.

Sec. 4. So far we have proved the existence of Q_s-indicators (i.e. indicators for strong segments). The Theorem 1.1. on incompleteness of PA requires Q_m-indicators (i.e. indicators for models of PA). Thus, we have to show yet the following.

Theorem 4.1. Each Q_s-indicator is also a Q_m-indicator.

The proof splits in several lemmas, which are of independent interest. First we prove.

Lemma 4.2. If $I \underset{e}{\subseteq} M$, (where M is a model) is strong in M, then I is also a model.

Proof. First, we show, that I is a stubstructure i.e. it is closed under addition and product. Suppose, for contradiction, that for some $u, v \in I$ we have $u + v \notin I$. Define $f(x) = u + x$, for $x \in M$. Thus, f is definable over M and since I is by assumption strong in M, there is an $e > I$ with the property (1). Since $f(v) = u + v \notin I$, we obtain that $u + v > e$, and hence there is an $x < v$ such that $u + x = e$. But $x \in I$, for $x < v \in I$, $u + x = f(x) = e$, a contradiction. Similarly, assume $u \cdot v \notin I$, for some $u, v \in I$. Puting $g(x) = u \cdot x$, we find a corresponding e, and infer, that $g(v) = u \cdot v > e$. There is an $x < v$ such that $u \cdot x \leqslant e$ and $u(x + 1) > e$. From $g(x) = u \cdot x \leqslant e$ follows $u \cdot x \in I$ and hence $u \cdot x + u = u(x + 1) \in I$, because I is already proved to be closed under addition. But $u(x + 1) \in I$ contradicts $u(x + 1) > e$.

The above implies now, that I is closed under the pairing function j and more generally under all function j_n enumerating the n-tuples. This, in turn, implies that the property (I) holds also for definable functions of several variables. Indeed, if f is a function of n variables definable over M, we put $g(x) = f(k_1^n(x), \ldots, k_n^n(x))$ (where k_i^n's are the converses of j_n). Now, g is definable and we find a corresponding e such that (1) holds. This e is good for f as well. Indeed, if $x_1, \ldots, x_n \in I$, then $j_n(x_1, \ldots, x_n) \in I$ and hence

$$g(j_n(x_1 \ldots x_n))$$

is either in I or else $> e$. But

$$g(j_n(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n)$$

and our claim is proved.

The next property of I is the following. For each formula $\phi(x_1 \ldots x_n)$ there is a formula $\overline{\phi}(x_1 \ldots x_n, y_1 \ldots y_m)$ and elements $e_1 \ldots e_m \in M$ such that for all $a_1 \ldots a_n \in I$ we have

$$I \models \phi\,[a_1, \ldots, a_n] \text{ iff } M \models \overline{\phi}[a_1, \ldots, a_n, e_1, \ldots, e_m] \quad \text{(eq)}$$

We prove this by induction on the length of ϕ: we put $\overline{\phi} = \phi$, for atomic ϕ and $\overline{\neg\phi} = \neg\overline{\phi}$ and $\overline{\phi \vee \psi} = \overline{\phi} \vee \overline{\psi}$. The verification of (eq) is immediate. Now, we treat the quantifier case. Assume inductively, that for a given $\phi(x, x_1, \ldots x_n)$ we have $\overline{\phi}(x, x_1, \ldots x_n, y_1 \ldots y_m)$ and $e_1 \ldots e_m$ so that (eq) holds and consider the formula $\exists x\phi$. Let $f(x_1 \ldots x_n)$ be the Skolem function for $\overline{\phi}$ i.e. the function defined thus

$$f(a_1, \ldots, a_n) = \begin{cases} \text{the smalles } a \text{ such that } M \models \overline{\phi}\,[a, a_1 \ldots a_n, e_1 \ldots e_m] \\ 0, \text{ if such an a does not exist.} \end{cases}$$

The function f is definable over M and by the previous claim we find an $e > I$ such that (1) holds. Let y_{m+1} be a variable not occuring in ϕ and put $\overline{\exists x\phi} = \exists x < y_{m+1}\, \overline{\phi}$ and $e_{m+1} = e$.

To prove (eq), assume $I \models \exists x\, \phi[a_1, \ldots, a_n]$. Hence, for some $a \in I$, we have $I \models \phi[a, a_1, \ldots, a_n]$ and, by inductive assumption, $M \models \overline{\phi}[a, a_1, \ldots, a_n, e_1 \ldots e_m]$. Since

$$a \in I < e = e_{m+1},$$

we obtain

$$M \models \exists\, x < e_{m+1}\, \overline{\phi}[a_1, \ldots, a_n, e_1, \ldots, e_m]$$

i.e.

$$M \models \overline{\exists\, x\, \phi}[a_1, \ldots, a_n, e_1, \ldots, e_m, e_{m+1}].$$

Conversely, if $\overline{\exists x\phi}$ is satisfied in M, then

$$a = f(a_1, \ldots, a_n) < e_{m+1} = e$$

and hence $a \in I$. By inductive assumption, $I \models \phi\,[a, a_1, \ldots, a_n]$ follows.

Now, since I is a substructure of M all the axioms 1-6 hold true in I. As remarked in Sec. 0., the schema (Ind) is equivalent to (Min), given axioms 1-6 alone. Hence, it suffices to show, that $I \models$ Min holds. Suposse, that $\phi(x, a_1, \ldots, a_n)$ defines in I a nonempty set. Since M is a model, then $M \models$ Min and we find the smallest $u_0 \in M$ such that

$$M \models \overline{\phi}[u_0, a_1 \ldots a_n, e_1 \ldots e_m].$$

Then, by (eq) we have $I \models \phi[u_0, a_1, \ldots, a_n]$. If there was an $u < u_0$ with $I \models \phi[u, a_1 \ldots a_n]$, then again by (eq) we had $M \models \overline{\phi}[u, a_1 \ldots a_n, e_1 \ldots e_m]$, a contradiction. Thus, u_0 is the smallest element satisfying ϕ in I and the Lemma is proved.

To prove the Theorem 4.1. we have to show that, for a Q_s-indicator Y, the equivalence in (ii) of the Definition of Sec. 1. holds. We can already show the "→" implication: assume $Y^M(a, b) > \omega$. Since, Y is Q_s-indicator, (ii) holds and hence there is an $I \subseteq_e M$, strong in M and $a \in I < b$. By the Lemma 4.2. $I \models PA$ and hence the right hand side of equivalence of (ii) is true.

The "←" implication will be proved in next section.

Sec. 5. Before completing the proof of 4.1., we wish to formulate an arithmetical counterpart of Ramsey's theorem. For a set $X \subseteq \omega$ we denote

$$[X]^n = \{<x_1 \ldots x_n> : x_1, \ldots, x_n \in X \text{ and } x_1 < \ldots < x_n\}.$$

Given a partition $[X]^n = A_0 \cup A_1$, with $A_0 \cap A_1 = \phi$, into two parts, a set $Z \subseteq X$ is said to be homogenous (for this partition) if either $[Z]^n \subseteq A_0$ or $[Z]^n \subseteq A_1$. Ramsey's theorem states, that if X is infinite, then for all n and arbitrary partition of $[X]^n$ in two parts, there exists an infinite homogenous set Z. The proof (see e.g. Kleinberg, 1973) uses induction and one sees, that if both X and A_0 are PA-definable, so is a homogenous set. Hence, replacing sets by defining formulas, one can formalize and prove this definable version of Ramsey's theorem in Peano arithmetic. To do this, let us denote for a given formula ϕ

$$V_n^{\phi}(x) : \bigwedge_{1 \leqslant j \leqslant n} \phi(k_j^n(x)) \ \& \bigwedge_{1 \leqslant j < n} (k_j^n(x) < k_{j+1}(x))$$

Thus, V_n^{ϕ} describes the increasing n-triples with terms in ϕ. An arbitrary formula $\psi(x)$ determines then a partition of V_n^{ϕ} (namely, $V_n^{\phi}(x) \ \& \ \psi(x)$ and $V_n^{\phi}(x) \ \& \ \neg\psi(x)$).

For a formula $\chi(x)$ we denote $\mathrm{Hom}(\chi, \phi, \psi, n)$:

$$\forall x\, [\chi(x) \rightarrow \phi(x)] \ \& \ \{\forall x\, [V_n^{\chi}(x) \rightarrow$$
$$\rightarrow \psi(x)] \vee \forall x\, [V_n^{\chi}(x) \rightarrow \neg\psi(x)]\}$$

Finally, $\exists|^{\infty} x\, \phi(x)$ stands for $\forall y\, \exists x\, [x > y \ \& \ \phi(x)]$. Now, Ramsey's theorem can be stated as follows: for arbitrary formulas ϕ and ψ and all $n \in \omega$ there is a formula χ such that the following holds

$$PA \vdash \exists|^{\infty} x\, \phi(x) \rightarrow \exists|^{\infty} x\, \chi(x) \ \& \ \mathrm{Hom}(\chi, \phi, \psi, n)$$

We shall use this fact in the proof of the following theorem.

Theorem 5.1. Let M be a countable model of PA. There exists a proper countable elementary extension K of M such that $M \subseteq_e K$ and M is strong in K.

Proof. We define $K = \mathrm{Def}(M^M)/F$, the definable ultrapower, where $\mathrm{Def}(M^M)$ denotes the family of definable over M functions $f: M \rightarrow M$ and F is an ultrafilter in the Boolean algebra of definable subsects of M. It is easy to see that Lös theorem reamins valid and hence the embedding of M into K by constant functions is elementary. Thus, K is a countable elementary extension of M and, if F is nonprinciple, the extension is proper. To satisfy $M \subseteq_e K$ and M to be strong in K we know to choose F more carefully.

Namely, we shall construct F in such a way that the following conditions will be satisfied

1. If $f \in \mathrm{Def}(M^M)$ and $f(x) \leqslant a$, for $x \in M$, then f is constant on a set $X \in F$.

2. If $f \in \mathrm{Def}(M^M)$ is such that for all $b \in M$ we have $M \models \mathrm{Fnc}[f(b), b]$, then there is an $X \in F$ such that for all $a < b < c < d$ from X we have

$$(\phi) \quad \forall i < a\ \{[f(c)(i) < b \vee f(d)(i) < b] \rightarrow f(c)(i) = f(d)(i)\}$$

To do this, enumerate all functions $f \in \mathrm{Def}(M^M)$, which are bounded and all those f from $\mathrm{Def}(M^M)$ which satisfy $M \models \mathrm{Fnc}\, [f^{(b)}, b]$, for all $b \in M$. Starting from $X_0 = M$ define inductively a decreasing sequence X_n of unbounded definable sets as follows: suppose X_{2n} is already defined. Take the n-th bounded function f (in the fixed enumeration). By Dirichlet's Principle f is constant on some definable, unbounded subset $X \subseteq X_{2n}$. Put $X_{2n+1} = X$. Now, take the n-th function f satisfying the premise of second condition and a partition $[X_{2n+1}]^4$ into A_0, A_1, where A_0 consists of those $a < b < c < d$ for which (ϕ) holds true and A_1 of the remaining ones. Since both S_{2n+1} and A_0 are definable and X_{2n+1} is unbounded we apply Ramsey's theorema to find a definable, unbounded set X, which is homogeneous for the partition. We have to show, that $[X]^4 \subseteq A_0$. Suposse, for contradiction, that $[X]^4 \subseteq A_1$. Fix $a < b$ for X. Then, for any $c < d$ from X, with $c > b$ there corresponds an $i < a$ such that

$$[f(c)(i) < b \vee f(d)(i) < b] \ \& \ [f(c)(i) \neq f(d)(i)]$$

By Dirichlet's Principle we find an $i_0 < a$, so that there arbitrarily high $c < d$ from X to which i_0 corresponds. Enumerating them, we find a definable function c such that $f(c(x)) < b$, for all x, and yet $x \neq y$ implies

$$f(c(x))\,(i_0) \neq f((c(y))\,(i_0),$$

which is impossible. Thus $[X]^4 \subseteq A_0$ and we put $X_{2n+2} = X$.

Now, let $F = \{X : \exists n [X_n \subseteq X]\}$, F is a filter and in fact, an ultrafilter. Indeed, if X is a definable subset of M, then its characteristic function

$$f(x) = \begin{cases} 0, & x \in X \\ 1, & x \in X \end{cases}$$

is bounded and by 1. either $X = f^{-1}(0)$ or $M \setminus X = f^{-1}(1)$ is in F. Since all sets in F are unbounded, F is nonprinciple. Also 1. implies, that $M \underset{e}{\subseteq} K$: if $f/F \leqslant a/F$, where $a/F \in M$ i.e. a is a constant, then $f(x) \leqslant a$, for $x \in X \in F$. There is an f' such that $f'/F = f/F$ and $f'(x) \leqslant a$, everywhere. By 1. $f'(x) = c \leqslant a$, for $x \in X' \in F$ and hence $f/F \in M$.

Finally, we prove, that M is strong in K. We shall use Lemma 3.2. We have $\mathrm{id}/F > M$ (id is the identity $\mathrm{id}(x) = x$ for all x), since F is nonprinciple. Let f be such that

$$K \models \mathrm{Fnc}\,[f/F, \mathrm{id}/F]$$

We have to find a function e such that for $i \in M$, either $f/F(i) \in M$ or else is $> e/F$. By Lös' theorem we have

$$M \models \mathrm{Fnc}[f(b), b],$$

for b from a set from F and we may assume that this holds that for all $b \in M$. By 2. we find and $X \in F$ such that for $a < b < c < d$ from X the formula (ϕ) holds. Let $i \in M$. Fix $u, v \in X$ such that $i < u < v$. There are two cases

a) There is a $c > v$ and there are $i < a < b < c$ (all from X) such that $f(c)(i) < b$. Then from all $d > c$ from X we have

$$f(c)(i) < b \vee f(d)(i) < b$$

and hence $f(d)(i) = f(c)(i)$, for all $d > c$ from X, which implies $f/F(i) = f(c)(i) \in M$ in K.

b) For all $c > v$ and all a, b such that $a, b, c \in X$ and $i < a < b < c$ we have $f(c)(i) \geqslant b$.

We put then $e(c) = \max\,\{b : b \in X \,\&\, b < c\}$, for all such c as above.

We have $e(c) \leqslant f(c)(i)$, for all $c \in X$ and $c > v$, and hence $f/F(i) \geqslant e/F$ in K. Note, that the function e does not depend on i. It remains to show, that $e/F > M$. If $e/F \in M$, then e is almost constant i.e. $e(x) = w$, for x from a set $Z \in F$. Then $Z \cap X \in F$ let $x_0 \in Z \cap X$ and $x_0 > w$ then for all $x \in Z \cap X$ and $x > x_0$ we have $e(x) > w$, a contradiction. Thus, $e/F > M$ and the proof is finished.

Corollary 5.2. Let M be a countable, nonstandard model. For arbitrary $a \in M$, there is an $I \underset{e}{\subseteq} M$, strong in M and such that $a \in I$.

Proof. Let K be an elementary extension of M as in Theorem 5.1 and let Y be a Q_s-indicator. Since M is strong in K we have $Y^K(a, b) > \omega$, for arbitrary $b \in K \setminus M$. Let $c \in M$ be such that $Y^K(a, b) > c > \omega$. Then we have

$$K \models \exists y [Y(a, y) > c]$$

and by elementarity we obtain $M \models \exists y\, [Y(a, y) > c]$, which gives $Y^M(a, b) > \omega$, for some $b \in M$. Hence, there is a segment I of M containing a.

Now, we finish the proof of Theorem 4.1. It remains to show the implication "$\leftarrow$" of (ii). Assume, then, there is a segment $M_1 \underset{e}{\subseteq} M$, such that $M_1 \models PA$ and $a \in M_1 < b$. By Corollary 5.2 there is an $I \underset{e}{\subseteq} M_1$, strong in M_1 and such that $a \in I \underset{e}{\subseteq} M_1 < b$. By Corollary 3.3. I is strong in M, as well. Since Y is a Q_s-indicator, we obtain $Y^M(a, b) > \omega$, which finishes the proof.

Let us note that it is possible to construct a Q_m-indicator directly. More generally if Γ is a theory extending PA and $\omega \models \Gamma$ and Γ is Σ_1-definable, then one can construct directly an indicator for models of Γ. The method is similar as in the proof of Theorem 3.4. but the game is more complicated.

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