

On the transformation of torques between the laboratory and center of mass reference frames

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It is commonly stated in Newtonian Mechanics that the torque with respect to the laboratory frame is equal to the torque with respect to the center of mass frame plus an $\mathbf{R} \times \mathbf{F}$ factor, with \mathbf{R} being the position of the center of mass and \mathbf{F} denoting the total external force. Although this assertion is true, there is a subtlety in the demonstration that is overlooked in the textbooks. In addition, it is necessary to point out that if the reference frame attached to the center of mass rotates with respect to certain inertial frame, the assertion is no longer true.

Keywords: Torque; center of mass; transformation of torques; fictitious forces.

En la mecánica newtoniana, es un hecho bien establecido que el torque con respecto al laboratorio es igual al torque visto por el centro de masa más un factor $\mathbf{R} \times \mathbf{F}$, donde \mathbf{R} denota la posición del centro de masa y \mathbf{F} la fuerza total externa. Aunque esta aseveración es correcta, hay una sutileza en la demostración que es pasada por alto en los textos tradicionales. Adicionalmente, es necesario clarificar que si el sistema de referencia asociado al centro de masa está en rotación con respecto a algún sistema inercial, dicha afirmación deja de ser cierta.

Descriptores: Torque; centro de masa; transformación de torques; fuerzas ficticias.

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In Newtonian Mechanics, we define the total external torque of a system of n particles (with respect to the laboratory frame that we shall assume as an inertial reference frame) as

$$\mathbf{N}_{\text{ext}} = \sum_{i=1}^n \mathbf{r}_i \times \mathbf{F}_{i(e)},$$

where \mathbf{r}_i , $\mathbf{F}_{i(e)}$ denote the position and total external force for the i th particle. The relation between the position coordinates between the laboratory (L) and center of mass (CM) reference frames^{*i*} is given by

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R},$$

with \mathbf{R} denoting the position of the CM about the L, and \mathbf{r}'_i denoting the position of the i -th particle with respect to the CM, in general the prime notation denotes variables measured with respect to the CM. An standard demonstration shows that [1]

$$\mathbf{N}_{\text{ext}} = \left(\sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}_{i(e)} \right) + \mathbf{R} \times \mathbf{F}, \tag{1}$$

where \mathbf{F} corresponds to the total external force on the system of particles (measured by the L). It is usually said that the first term on the right side of Eq. (1) provides the external torque relative to the CM. Strictly speaking, this is not the case, since $\mathbf{F}_{i(e)}$ is a force measured with respect to the L system; and since the CM is not in general an inertial reference frame, the force measured by the CM is not equal to the force in the L system. As is well known from the theory of non-inertial systems [2], the total force on the i th particle

measured about the CM reads

$$\mathbf{F}'_i = \mathbf{F}_i - m_i \mathbf{A}_{\text{CM}}, \tag{2}$$

where \mathbf{A}_{CM} denotes the acceleration of the CM with respect to the L. Taking into account that the force on the i th particle is given by the sum of the external forces plus the internal ones, we have

$$\mathbf{F}'_{i(e)} + \sum_{k=1}^n \mathbf{F}'_{ik} = \mathbf{F}_{i(e)} + \sum_{k=1}^n \mathbf{F}_{ik} - m_i \mathbf{A}_{\text{CM}}, \tag{3}$$

where \mathbf{F}_{ik} denotes the internal force on the i th particle due to the k th particle, and m_i is the corresponding mass. Now, if we take into account that the internal forces are independent of the reference frame^{*ii*}, and using Eq. (3), we get

$$\mathbf{F}'_{i(e)} = \mathbf{F}_{i(e)} - m_i \mathbf{A}_{\text{CM}}. \tag{4}$$

From Eq. (4), the external torque about the CM becomes

$$\begin{aligned} \mathbf{N}_{\text{CM}} &= \sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}'_{i(e)} = \sum_{i=1}^n \mathbf{r}'_i \times [\mathbf{F}_{i(e)} - m_i \mathbf{A}_{\text{CM}}] \\ &= \sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}_{i(e)} - M \left(\frac{1}{M} \sum_{i=1}^n m_i \mathbf{r}'_i \right) \times \mathbf{A}_{\text{CM}}, \end{aligned}$$

where the term in parentheses corresponds to the position of the CM with respect to the CM itself; therefore it clearly vanishes, from which we see that

$$\sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}_{i(e)} = \sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}'_{i(e)} = \mathbf{N}_{\text{CM}}, \tag{5}$$

and replacing Eq. (5) in Eq. (1), we get

$$\mathbf{N}_{\text{ext}} = \left(\sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}'_{i(\text{ext})} \right) + \mathbf{R} \times \mathbf{F}. \quad (6)$$

From Eq. (6) we can assert that *the total external torque about the laboratory is equal to the external torque about the center of mass plus the torque equivalent to a particle located at the position of the center of mass undergoing the total external force of the system of particles.*

This coincides with the assertion given in common texts. However, this statement follows from Eq. (6) and not from Eq. (1), as appears in the literature. Moreover, as it is clear from the development above, the demonstration of Eq. (5) is the clue to making this statement. In turn, Eq. (5) is satisfied because the fictitious forces do not contribute to the total external torque.

Finally, this clarification is also necessary in order to establish the equation

$$\frac{d\mathbf{L}_{\text{CM}}}{dt} = \mathbf{N}_{\text{CM}},$$

with \mathbf{L}_{CM} denoting the total angular momentum of the system of particles about the CM. As is well known, this equation is valid even if the center of mass is a non-inertial system of reference [1], in which case the fictitious forces should be included in the demonstration.

Example 1 *Let us consider the case of a yo-yo of mass M and radius b which drops, unwinding by means of a string, as shown in Fig. 1a. In a system of reference located at O , the forces over the yo-yo are the weight and the tension, both of which produces a torque, and the problem is quite complicated. In the system of reference of the CM, we have an additional fictitious force. The weight has null torque, the tension gives a torque of $bT\hat{\mathbf{k}}$, and the torque associated with the fictitious force can be written as*

$$\mathbf{N}_{\text{fict}} = - \int dm \mathbf{r}' \times \mathbf{a}_{\text{CM}},$$

where \mathbf{a}_{CM} denotes the acceleration of the CM with respect to O . Factorizing \mathbf{a}_{CM} , we get $\mathbf{N}_{\text{fict}} = -M\mathbf{R}' \times \mathbf{a}_{\text{CM}} = 0$, and hence the fictitious torque vanishes. Therefore we get the result usually found in textbooks

$$Mg - T - Ma = 0, \quad \frac{d\mathbf{L}_{\text{CM}}}{dt} = I \frac{d\omega}{dt} = bT\hat{\mathbf{k}}.$$

Although the result is the same as in the literature, the correct procedure requires taking into account the fact that the system of reference attached to the CM is non-inertial.

In the case in which the acceleration of the center of mass is constant, we can relate the fact that the fictitious forces do not produce torque, to the principle of equivalence. According to this principle, the resultant fictitious force on the system of particles is equivalent to a weight due to a uniform gravitational field. Consequently, it appears as a gravity force acting on the CM (see Sec. 8.4 in Ref. 2), and therefore the

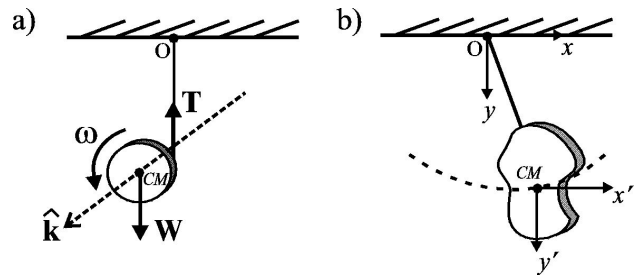


FIGURE 1. (a) A yo-yo unwinding from a string. Its CM is accelerated with respect to an inertial reference frame. (b) A physical pendulum hanging from a fixed point O . The system attached to O is considered inertial, while the system associated with the CM is in relative translation with respect to the latter. In this case, the acceleration of the CM is not constant, but still the fictitious force associated with the CM system does not give a torque.

torque with respect to the CM is clearly zero. Nevertheless, we point out that, according to our demonstration, the torque due to the fictitious forces vanishes even if the acceleration \mathbf{A}_{CM} is not constant, in which case the equivalence principle does not hold. Fig. 2b shows a system in which the CM reference frame is in relative translation with respect to an inertial frame; in this case, the torque associated with the fictitious force vanishes, although \mathbf{A}_{CM} is not constant and the equivalence principle cannot be applied.

In the previous treatment, we have implicitly assumed that the center of mass reference frame defines coordinate axes that are only in relative translation with respect to the inertial system. Now let us assume that the center of mass is at rest with respect to a certain inertial frame, but that the axes of coordinates associated with the CM defines a rotating system with angular velocity Ω with respect to the coordinate axes of the inertial frame. As is well known, the vectors of position coincide in both systems of reference (*i.e.* $\mathbf{r}'_i = \mathbf{r}_i$), but not their time derivatives. The total torque on the system is

$$\mathbf{N}'_{\text{total}} = \sum_{i=1}^n \mathbf{r}_i \times \mathbf{F}'_i = \sum_{i=1}^n \mathbf{r}_i \times \left[\sum_{i \neq k}^n \mathbf{F}_{ik} + \mathbf{F}_{i(e)} - 2m_i \Omega \times \mathbf{v}'_i - m_i \Omega \times (\Omega \times \mathbf{r}_i) \right] \quad (7)$$

where $\sum_{i \neq k}^n \mathbf{F}_{ik} + \mathbf{F}_{i(e)}$ defines the force on the i th particle, measured by the inertial reference frame, and $-2m_i \Omega \times \mathbf{v}'_i - m_i \Omega \times (\Omega \times \mathbf{r}_i)$ refers to the coriolis and centrifugal forces respectively. For simplicity, we shall assume that the internal forces are central and that they depend on relative positions only (and not on relative velocities or higher time derivatives), so that the torque associated with the internal forces vanishes, and only the external torque contributes, yielding

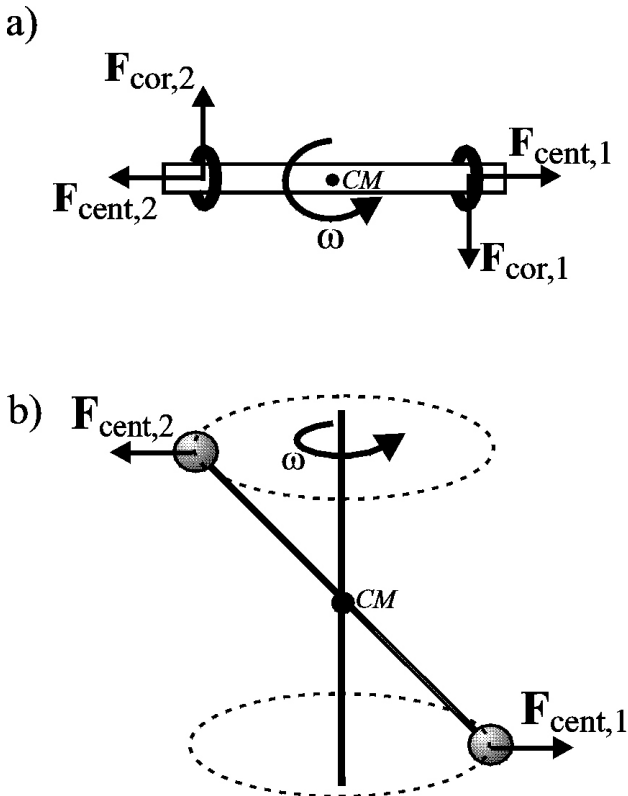


FIGURE 2. (a) Aerial view of a couple of identical beads sliding along a rotating wire. The Z -axis is off of the paper. The centrifugal and coriolis forces are indicated in the diagram. (b) Two identical particles joined by a thin rod of negligible mass, and rotating at constant angular velocity ω ; the centrifugal forces are indicated in the diagram, and no coriolis force appears in a system that rotates with the particles. In both diagrams we show only the fictitious forces.

$$\begin{aligned} \mathbf{N}'_{\text{ext}} &= \sum_{i=1}^n \mathbf{r}_i \times [\mathbf{F}_{i(e)} - 2m_i\boldsymbol{\Omega} \times \mathbf{v}'_i - m_i\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_i)] \\ &= \mathbf{N}_{\text{ext}} + \mathbf{N}_{\text{fict}} \end{aligned}$$

with

$$\mathbf{N}_{\text{fict}} = - \sum_{i=1}^n \{m_i \mathbf{r}_i \times [2\boldsymbol{\Omega} \times \mathbf{v}'_i + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_i)]\} \quad (8)$$

Eq. (8) shows that in the most general case, the fictitious forces could produce a torque. Let us illustrate this point with the following example.

Example 2 A couple of identical beads are sliding without friction along a rigid wire (of negligible mass) rotating at constant angular speed ω (see Fig. 2a). In this case, the CM of the system is at rest with respect to an inertial frame. But we can define a rotating system with the origin at the CM and rotating with the wire. The coriolis and centrifugal forces on each bead are plotted in Fig. 2a. Since we are interested in the torque produced by fictitious forces only, we do not consider the torque coming from the (real) normal force. Assuming the axis of rotation to be along with the z -axis, we obtain

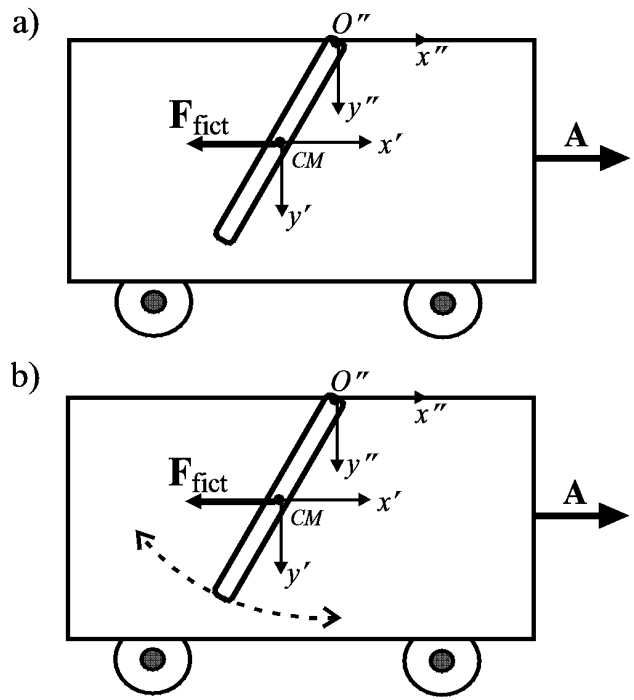


FIGURE 3. In both figures, the car has an acceleration \mathbf{A} and we assume 2 dimensional reference frames for simplicity: (a) The thin stick is at rest with respect to the car. The acceleration of the CM of the thin stick is precisely \mathbf{A} , and the system of reference S'' has the same acceleration. However the origin of S'' does not lie at the CM of the stick. It can be shown that, in this case, the fictitious force produces a torque. (b) The stick is oscillating around its equilibrium position. In this case, the acceleration of S'' does not coincide with the acceleration of the center of mass of the stick. Once again, there is a torque associated with the fictitious force.

$$\mathbf{N}_{\text{fict}} = \mathbf{r}_1 \times (\mathbf{F}_{\text{cor},1} + \mathbf{F}_{\text{cent},1}) + \mathbf{r}_2 \times (\mathbf{F}_{\text{cor},2} + \mathbf{F}_{\text{cent},2}).$$

The centrifugal forces are parallel to the corresponding vectors of position so that their contribution vanish, we also see that $\mathbf{r}_2 = -\mathbf{r}_1$ and $\mathbf{F}_{\text{cor},2} = -\mathbf{F}_{\text{cor},1}$ and the fictitious torque becomes

$$\mathbf{N}_{\text{fict}} = 2\mathbf{r}_1 \times \mathbf{F}_{\text{cor},1} = -2r_1\omega v'_1 \hat{\mathbf{k}} = -2r_1 \dot{r}_1 \omega \hat{\mathbf{k}}.$$

The values of r_1, \dot{r}_1 can be calculated to be (see example 8.7 in Ref. [2])

$$r_1 = Ae^{\omega t} + Be^{-\omega t} ; \dot{r}_1 = \omega (Ae^{\omega t} - Be^{-\omega t}),$$

and the fictitious torques do not vanish. It is not difficult to find examples in which the centrifugal force could also contribute to the fictitious torque. The reader could try with the system described by Fig 2b.

In conclusion, we can state that

$$\mathbf{N}_{\text{ext}} = \mathbf{N}_{\text{CM}} + \mathbf{R} \times \mathbf{F}_{\text{ext}}; \quad \frac{d\mathbf{L}_{\text{CM}}}{dt} = \mathbf{N}_{\text{CM}}$$

only if the system of reference attached to the CM is not in relative rotation with respect to a certain inertial frame. It is

related to the fact that the fictitious forces produced by relative accelerated translation do not produce torque, while the fictitious forces associated with relative rotation do.

A final comment about a more general framework; let us assume a non-inertial system (denoted by S'') in relative translation with respect to an inertial frame. If the acceleration of S'' coincides with A_{CM} , but the origin of S'' is not

located at the CM, the fictitious forces associated with S'' **do produce a torque**; Fig. 3a shows an example. In addition, if the acceleration of S'' is not the same as the acceleration of the CM, the fictitious forces associated with S'' also produce a torque, as shown in Fig. 3b. These fictitious torques can be calculated by assuming that the resultant fictitious force is applied to the CM even if A_{CM} depends on time.

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- i* For the center of mass reference frame we mean a system of reference whose origin lies at the CM, at that is in relative translation (but not in relative rotation) with respect to certain inertial frame.
 - ii* In the case of central internal forces, it is clear since they are functions of the relative positions among different pairs of particles. Nevertheless, even in the case in which the forces are not central, they depend on relative positions, relative velocities,

relative accelerations etc., so that the invariance under different frames still holds.

1. M. Alonso and E. Finn, *Fundamental University Physics, Vol I, Mechanics* (Addison-Wesley Publishing Co., Massachusetts, 1967), Example 9.7.
2. D. Kleppner and R. Kolenkow, *An introduction to mechanics* (McGRAW-HILL KOGAKUSHA LTD, 1973), Sections 8.3 and 8.5.