Addendum to "On the vector solutions of Maxwell equations in spherical coordinate systems"

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The discussion of our previous work concerning the vector solution of boundary-value problems in electromagnetism is extended to the case of no azimuthal symmetry by means of the spin-weighted spherical harmonics.

Keywords: Maxwell equations; electric and magnetic fields; boundary-value problems; spin-weighted spherical harmonics.

Se extiende la discusión de nuestro trabajo anterior sobre la solución vectorial de problemas con valores de frontera en electromagnetismo al caso sin simetría azimutal mediante el uso de los armónicos esféricos con peso de espín.

Descriptores: Ecuaciones de Maxwell; campos eléctrico y magnético; problemas con valores de frontera; armónicos esféricos con peso de espín.

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In a recent paper [1], we introduced a somewhat different approach to solving boundary-value problems in spherical coordinates for time-independent and time-dependent electric and magnetic fields, without involving the scalar and vector potentials. We showed that the process includes the same mathematics of separation of variables as the usual approach of solving for potentials. However, it is restricted by the assumption of azimuthal symmetry. We now wish to remove this constraint. So the natural complete orthonormal set of expansion functions to consider for the vector solutions of Maxwell equations are the spin-weighted spherical harmonics [2–5]. The purpose of this note is to recast the general vector solutions for the electric and magnetic fields, and illustrate their applications on boundary-value problems by dealing, for simplicity's sake, with the commonplace examples solved in Ref. 1.

For time-independent electric and magnetic phenomena, the fields outside sources satisfy the vector Laplace equation $\nabla^2\mathbf{F}=\mathbf{0}$ with the subsidiary condition $\nabla\cdot\mathbf{F}=0$. The general solution in terms of the spin-weighted spherical harmonics ${}_sY_{lm}$ [4], with spin weight $s=0,\pm1$ and ${}_0Y_{lm}=Y_{lm}$, can be written as

$$F_0=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{lm}r^{l-1}+\frac{b_{lm}}{r^{l+2}}\right)Y_{lm},$$

$$F_\pm=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}v_\pm\ {}_{\pm1}Y_{lm}, \tag{1}$$

with

$$v_\pm=c_{lm}r^l+\frac{d_{lm}}{r^{l+1}}$$
$$\mp\sqrt{\frac{l(l+1)}{2}}\left(\frac{a_{lm}}{l}\ r^{l-1}-\frac{b_{lm}}{l+1}\ \frac{1}{r^{l+2}}\right), \tag{2}$$

where $F_0=\mathbf{e}_0\cdot\mathbf{F}=F_r$, $F_\pm=\mathbf{e}_\pm\cdot\mathbf{F}=(F_\theta\pm iF_\varphi)/\sqrt{2}$ are the components of the field with spin weight $s=0,\pm1$, respectively, and $\mathbf{e}_0=\hat{\mathbf{r}}$, $\mathbf{e}_\pm=(\hat{\theta}\pm i\hat{\varphi})/\sqrt{2}$ are the spin-weighted combinations of the orthonormal basis.

For harmonic time-dependent sources and fields, the electric and magnetic fields in regions apart from sources satisfy the vector Helmholtz equation $\nabla^2\mathbf{F}+k^2\mathbf{F}=\mathbf{0}$ with the transverse condition $\nabla\cdot\mathbf{F}=0$. The general solution [2, 4] now becomes

$$F_0=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}\frac{j_l(kr)}{r}+b_{lm}\frac{n_l(kr)}{r}\right]Y_{lm},$$

$$F_\pm=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}w_\pm\ {}_{\pm1}Y_{lm}, \tag{3}$$

with

$$w_\pm=c_{lm}j_l(kr)+d_{lm}n_l(kr)$$
$$\mp\frac{a_{lm}}{\sqrt{2l(l+1)}}\frac{1}{r}\ \frac{d}{dr}[r\,j_l(kr)]$$
$$\mp\frac{b_{lm}}{\sqrt{2l(l+1)}}\frac{1}{r}\ \frac{d}{dr}[r\,n_l(kr)], \tag{4}$$

where the spherical Hankel functions, $h_l^{(1)}$ and $h_l^{(2)}$, instead of the spherical Bessel functions, j_l and n_l, may be required by boundary conditions.

We note that $F_+=F_-$ in the case of boundary-value problems having azimuthal symmetry with $F_\varphi=0$. This implies that $c_{lm}=d_{lm}=0$ in Eqs. (2) and (4), which leads in turn to the solutions obtained in Ref. 1.

The boundary conditions for the electric and magnetic fields must be expressed in terms of their spin-weighted components. Assuming that the boundary surface is a sphere with

$\mathbf{n} = \mathbf{e}_0$, we obtain

$$D_{10} - D_{20} = \rho_S, \quad E_{1\pm} - E_{2\pm} = 0,$$
$$B_{10} - B_{20} = 0, \quad H_{1\pm} - H_{2\pm} = \mp i J_{S\pm}. \tag{5}$$

To illustrate the use of the above formulas for static fields, we choose the example of the electric field due to a ring having radius a with total charge Q uniformly distributed and lying in the x-y plane, which is also worked out in Ref. 1. The surface charge density on $r = a$, localized at $\theta = \pi/2$, is

$$\rho_S = \frac{Q}{2\pi a^2}\, \delta(\cos\,\theta), \tag{6}$$

which can be expanded using the series representation of the Dirac delta function in terms of spherical harmonics

$$\delta(\cos\,\theta) = 2\pi \sum_{l=0}^{\infty} \; Y_{l0}(\frac{\pi}{2}, 0)\; Y_{l0}(\theta, 0). \tag{7}$$

Taking into account the cylindrical symmetry of the system and the requirement that the series solutions in Eqs. (1) and (2) be finite at the origin, vanish at infinity and satisfy the boundary conditions of Eq. (5) at $r = a$ for all values of the angle θ, namely, $E_\pm$ continuous at $r = a$ and E_0 discontinuous at $r = a$, it is found that the spin-weighted components of the electric field are given by

$$E_0 = \frac{Q}{\epsilon_\circ r^2} \sum_{l=0}^{\infty} \frac{1}{2l+1}\; Y_{l0}(\frac{\pi}{2}, 0)\; Y_{l0}(\theta, 0) \times \begin{cases} (l+1)\left(\dfrac{a}{r}\right)^l, & r > a \\ -l\left(\dfrac{r}{a}\right)^{l+1}, & r < a \end{cases} \tag{8}$$

$$E_\pm = \pm \frac{Q}{\epsilon_\circ r^2} \sum_{l=1}^{\infty} \frac{\sqrt{l(l+1)}}{\sqrt{2}(2l+1)}\; Y_{l0}(\frac{\pi}{2}, 0)\; {}_{\pm 1}Y_{l0}(\theta, 0) \times \begin{cases} \left(\dfrac{a}{r}\right)^l, & r > a \\ \left(\dfrac{r}{a}\right)^{l+1}, & r < a \end{cases} \tag{9}$$

Note that the discontinuity of the lth component of E_0 in Eq. (8) at $r = a$ is connected, according to Eq. (5), with the corresponding component of the surface charge density ρ_S obtained from Eqs. (6) and (7), exhibiting the unity of the multipole expansions of fields and sources.

As an example of time-varying fields, we consider the problem of the magnetic induction field from a current $I = I_\circ e^{-i\omega t}$ in a circular loop with radius a lying in the x-y plane. The surface current density on $r = a$ is

$$\mathbf{J}_S = \frac{I_\circ}{a}\, \delta(\cos\,\theta)\; e^{-i\omega t}\; \hat{\varphi}, \tag{10}$$

where for the delta function we now use the expansion

$$\delta(\cos\,\theta) = 2\pi \sum_{l=1}^{\infty} \; {}_{\pm 1}Y_{l0}(\frac{\pi}{2}, 0)\; {}_{\pm 1}Y_{l0}(\theta, 0). \tag{11}$$

The solution of the Helmholtz equation for the magnetic induction field in Eqs. (3) and (4), which is finite at the origin, represents outgoing waves at infinity and satisfies the boundary conditions of Eq. (5) at $r = a$ with $J_{S\pm} = \mathbf{e}_\pm \cdot \mathbf{J}_S$, becomes

$$B_0 = \pm i \frac{2\pi \mu_\circ I_\circ k a}{r} e^{-i\omega t} \sum_{l=1}^{\infty} \sqrt{l(l+1)}\; {}_{\pm 1}Y_{l0}(\frac{\pi}{2}, 0) \times Y_{l0}(\theta, 0) \begin{cases} j_l(ka)\; h_l^{(1)}(kr), & r > a \\ j_l(kr)\; h_l^{(1)}(ka), & r < a \end{cases} \tag{12}$$

$$B_\pm = -i \frac{2\pi \mu_\circ I_\circ k^2 a}{\sqrt{2}} e^{-i\omega t} \sum_{l=1}^{\infty} \; {}_{\pm 1}Y_{l0}(\frac{\pi}{2}, 0)\; {}_{\pm 1}Y_{l0}(\theta, 0) \times \begin{cases} j_l(ka)\left[h_{l-1}^{(1)}(kr) - \dfrac{l}{kr} h_l^{(1)}(kr)\right], & r > a \\ h_l^{(1)}(ka)\left[j_{l-1}(kr) - \dfrac{l}{kr} j_l(kr)\right], & r < a \end{cases} \tag{13}$$

The discontinuity of the lth component of $B_\pm$ in Eq. (13) at $r = a$ is connected, according to Eq. (5), with the lth component of the surface current density $J_{S\pm}$ deduced from Eqs. (10) and (11).

Finally, by using the expressions of the spin-weighted spherical harmonics in Eqs. (8), (9), (12) and (13), it is seen that the results in Ref. 1 are obtained.

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