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A PROOF OF THE DISCRETENESS OF THE BOSONIC MEMBRANE SPECTRUM

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Abstract

Using a spectral theorem for Schrödinger like equations, we present a proof of the discreteness of the spectrum for open bosonic membrane, considering $\mathbb{L}^2$ immersion maps from the 2 dimensional interval, at constant time, to flat minkowski space. The residual area preserving symmetry and the BRST action were studied in a residual gauge fixing choice that allows a simplification of the effective membrane action. Some differences between discretised membrane and the $SU(N \to \infty)$ Yang Mills model explains apparently contradictory results reported on the spectrum.

Keywords: String Theory, Quantization of singular systems, spectral theory.

UNA PRUEBA DE LA DISCRETITUD DEL ESPECTRO DE LA MEMBRANA BOSÓNICA ABIERTA

Resumen

Usando un teorema espectral para ecuaciones tipo Schrödinger, se presenta una prueba de la discretitud del espectro de autovalores de la membrana bosónica abierta, considerando inmersiones $\mathbb{L}^2$ del intervalo 2 dimensional, a tiempo constante, al espacio minkowskiano plano. Se estudian la simetría residual que preserva áreas y su acción BRST en una elección de calibre residual que permite simplificar la acción efectiva de la membrana. Algunas diferencias entre la membrana discretizada y el modelo Yang Mills con $SU(N \to \infty)$ explican resultados aparentemente contradictorios sobre el espectro.

Palabras clave: Teoría de cuerdas, Cuantización de sistemas singulares, Teoría espectral

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Introduction

String Theory is the most promising candidate for unification theory and the quantization of gravity. It is now evident that the theory involves several types of D-p-branes that arises from non perturbative quantization of the elusive M-theory [1]. One of the sectors is the minimally immersed supermembrane in 11 dimensions [2][3], due to this, the problem of the supermembrane spectrum comes out as a matter of intense research. Long ago, in [4] it was established that the open supermembrane was unstable, the proof considers a regularized SU(N) Super Yang-Mills 1-dimensional version of the theory that had a continuous spectrum coming from string spikes singularities. Several years later it had been reported [6][7] that by compactifying the spatial dimensions in an irreducible way on a torus, the spectrum renders to be discrete, but other reported results disagree [8][9]. It is our feeling that the discreteness-continuity question depends on the topological invariants as, for example, the irreducibility wrapping number in particular, or the presence of non trivial central charges [10].

In contrast with the large number of works devoted to the closed supermembranes, in several compactifications [11], matrix models[12], non-abelian cases [13][2], with fermionics contributions [12], etc., there exist relatively few works about the quantization of the open bosonic membrane [14][15] [16]. Although there is evidence that the closed membrane spectrum is discrete when wrapping over compact dimensions [17][6][7], there is still controversy about the discreteness of the spectrum of the bosonic membrane [8][9], specially for the open membrane case [4].

In String theory, it is important to know if all the (solitonic) sectors, including the bosonic membrane, have a discrete spectrum, otherwise the theory will be not regularizable. Even in Landscape approach, the degeneration multiplicity of the ground state is giant, 10^{500}, but still finite [18]. So, the objective of this paper is to present a proof of the discreteness of the open bosonic membrane spectrum that could explain contradictory spectrum results. The proof was performed from a modified BFV [19] functional point of view.

The organization of this paper is the following: in section 2 we review the classical membrane theory, its invariances, constraints and residual gauge symmetry in the conformal light cone gauge fixing. In section 3 we explain the discretised membrane model and its symmetry generators. In section 4 we fix the residual gauge and explicitly construct the invariant residual BRST effective action. In section 5 we present a proof of the discreteness of the open bosonic membrane spectrum and explain apparently continuous spectrum. Finally in section 6 we present some conclusions and remarks.

Gauge Fixing and Residual Gauge Group

We will start from the usual second order action

$$S = -\frac{1}{8\pi^2\beta}\int_B \sqrt{-g}(g^{ab}x^{\mu},_a x_{\mu},_b - 1)d^3\sigma \tag{1}$$

which is equivalent to the Nambu-Goto-Dirac action [20] over the field equations.

The classical Hamiltonian and constraints can be obtained from (1), following Dirac's procedure and using the ADM parametrization:

$$\mathcal{H} = \frac{N}{2\sqrt{\gamma}}(p^2+\gamma) + N^i(px,_i) \tag{2}$$

where $\frac{N}{2\sqrt{\gamma}}$ and N^i are the Lagrange multipliers associated with the 3-d diffeomorphisms generating first class constraints:

$$\phi = \frac{1}{2}(p^2+\gamma)\ , \quad \phi_i = px,_i \tag{3}$$

The conformal gauge fixing for this theory is defined as in [21]

$$\begin{aligned} g_{oj} &= 0 \\ g_{oo} + \gamma &= 0. \end{aligned} \tag{4}$$

It was shown long time ago [22] that although this gauge fixes the Lagrange multipliers, there is still a residual gauge group. This residual gauge group has as parameters the solutions to the homogeneous equations that arise from the gauge invariance of the gauge fixing conditions, i.e. the solutions to:

$$\begin{aligned} \delta(g_{oo}+\gamma) &= 2[-\partial_o\zeta^o + \partial_i\zeta^i] = 0. \\ \delta(g_{oi}) &= \gamma_{ij}\partial_o\zeta^i + \gamma\partial_i\zeta^o = 0. \end{aligned} \tag{5}$$

In the conformal gauge, for the initial condition $\partial_o x^+(\tau=0) = c^+$, the time evolution of x^+ is given by the field equation: $\partial_{oo}x^+ = 0$. That equation could be directly integrated as: $x^+ = c^+\tau - h(\sigma^1,\sigma^2)$ with an arbitrary addition function $h(\sigma^1,\sigma^2)$. This arbitrary function could be compensated using the residual gauge parameter $\zeta^o = \zeta^o_{res}(\tau,\sigma^1,\sigma^2)$. Note that this allow us to fix the light cone gauge (LCG)

$$x^+ = c^+\tau, \text{ because } x^+ + \delta x^+ = c^+\tau - h(\sigma^1,\sigma^2) + \partial_o\zeta^o_{res} = c^+\tau \tag{6}$$

fixing ζ^o as a particular solution to the equation $\zeta^o_{res} = \int d\tau\ h(\sigma^1,\sigma^2)$. Defining $\partial_o\zeta^i = \partial_o\zeta^i_{res} - \gamma^{ij}\gamma\partial_j\zeta^o_{res}$ then from (5)

$$\partial_o\zeta^i_{res} = 0,\ \ \partial_i\zeta^i_{res} = 0 \Rightarrow \zeta^i_{res} = \epsilon^{ij}\partial_j f(\sigma^1,\sigma^2).$$

The LCG (6) plus conformal gauge (4) fixing allows us to determine the $d-2$ transverse part, explicitly solving the constraints we get:

$$\partial_i x^- = \frac{1}{c^+}\partial_i\vec{x}.\vec{p}, \quad p^+ = c^+, \quad p^- = \partial_0 x^-. \tag{7}$$

These equations do not exhaust the content of the constraints (3). Indeed, if we take the *2d curl* of ϕ_i, we get the residual constraint [23]

$$T = \epsilon^{ij}\vec{p}_{,i}\,\vec{x}_{,j} = \vec{p}_{,1}\,.\vec{x}_{,2} - \vec{p}_{,2}\,.\vec{x}_{,1} \tag{8}$$

that generates the residual group. This group is the subgroup of 2 dimensional diffeomorphisms that preserve areas, and its generator has a closed first class algebra [15], namely:

$$\{T(\sigma), T(\bar\sigma)\}_{\mathcal{PB}} = \epsilon^{ij}\partial_i T(\sigma)\partial_j\delta(\sigma-\bar\sigma). \tag{9}$$

The action of this group on the canonical variables is given by the following Poisson brackets:

$$\delta_T x = \{x(\sigma), \int d^2\bar\sigma\lambda(\bar\sigma)T(\bar\sigma)\}_{\mathcal{PB}} = \xi^j\partial_j x(\sigma) \tag{10}$$

$$\delta_T p = \{p(\sigma), \int d^2\bar\sigma\lambda(\bar\sigma)T(\bar\sigma)\}_{\mathcal{PB}} = \partial_j(\xi^j p(\sigma)) \tag{11}$$

where $\xi^i \equiv \epsilon^{ij}\partial_j\lambda$ may be identified with ζ^i_{res}. According to the above formulas, the coordinates x transform as a scalars while their corresponding momenta p as scalar densities in two dimensions as expected.

Using the equations of motion that comes from the action (1) and the constraint and gauge fixing equations (3) and (4) we get

$$\partial_a x^- = -\partial_0 x^I\partial_a x^I \tag{12}$$
$$\partial_0 x^- = -\tfrac{1}{2}(\partial_0 x^I)^2 - \tfrac{1}{2}det(\partial_a x^I\partial_b x^I) \tag{13}$$

where we denoted by x^I the Light cone transverse part of x^m. These equations allow us to solve the minus sector x^- .

The equations for the d-2 transverse sector, may be obtained the following effective transverse action

$$L = \frac{1}{2}(\partial_0 x^I)^2 - \frac{1}{4}det(\partial_a x^I \partial_b x^I) \tag{14}$$

this could be rewritten as

$$L = \frac{1}{2}(\partial_0 x^I)^2 - \frac{1}{4}\{x^I, x^J\}_{\mathcal{LB}}\{x^I, x^J\}_{\mathcal{LB}} \tag{15}$$

where the Lie bracket is defined by

$$\{A, B\}_{\mathcal{LB}} \equiv \epsilon^{ij}\partial_i A \partial_j B \ . \tag{16}$$

The action (14) is invariant under gauge transformations generated by the constraint (3) but only when the parameters are $\xi^i = \epsilon^{ij}\partial_j \lambda$, the invariance of (15) by the Lie brackets follows from

$$\delta x(\sigma) = \{x(\sigma), \lambda(\sigma)\}_{\mathcal{LB}} \ , \tag{17}$$

that is equivalent to the residual gauge invariance generated by T through

$$\begin{aligned} \delta_T x &= \{x(\sigma), \int d^2\bar{\sigma}\lambda(\bar{\sigma})T(\bar{\sigma})\}_{\mathcal{PB}} \\ &= \{x(\sigma), \lambda(\sigma)\}_{\mathcal{LB}} = \zeta^j \partial_j x(\sigma) \end{aligned} \tag{18}$$

$$\text{where: } \zeta^j \equiv \epsilon^{ij}\partial_j \lambda(\sigma) \ . \tag{19}$$

The transverse action (15) has just the structure of the action for a Yang Mills theory reduced to one dimension, in the Coulomb gauge [4]. This equivalence is a particular characteristic of the 2-brane and could not be easily extended to other p-branes. Although we still have to discuss the residual gauge symmetry of the membrane generated by the residual constraint (8) that in principle is absent from a Yang Mills theory.

Discretized membrane model

Introducing a base of functions over the 2D open interval $I_2 = (\sigma^1, \sigma^2) = (0, 2\pi) \times (0, 2\pi)$ at constant time [4]. A complete basis of $L^2(I_2)$ immersions maps $x^{IA}(\tau, \sigma^1, \sigma^2)$ at constant time τ is the 2-dimensional Fourier basis

$$Y_A = Y_{mn} = exp(im\sigma_1 + in\sigma_2) \tag{20}$$

with an Algebra given by Lie Bracket

$$\{Y_A, Y_B\}_{\mathcal{LB}} = f_{ABC}\, Y_C \tag{21}$$
$$f^A_{BC} = f^{mn}_{pq,rs} = (B \times C)\delta^A_{B+C} \tag{22}$$

and structure constants f^A_{BC}, where $A = (m, n); B = (p, q)$ and $C = (r, s)$ with $A, B, C \in \mathbb{Z}^2$.

Both, momentum and position coordinates could be expanded in the Fourier basis as

$$x^I(\tau, \sigma^i) = x^I_o(\tau) + \sum_A x^{IA}(\tau) Y_A(\sigma^1, \sigma^2), \tag{23}$$
$$p^I(\tau, \sigma^i) = p^I_o(\tau) + \sum_A p^{IA}(\tau) Y_A(\sigma^1, \sigma^2), \tag{24}$$

where x^I_o and p^I_o corresponds to the center of mass coordinates and $A \in \mathbb{Z}^2 - (0, 0)$, and the Fourier expansion converges weakly for $L^2(I_2)$.

The Hamiltonian is obtained from (15) using $D_o x^I = \partial_o x^I = p^I$, is expanded as:

$$H = \frac{1}{2c^+}[p^I_o p^I_o + p^{IA} p^{IA}] + \frac{1}{4}[f_{ABC} x^{IB} x^{JC}]^2, \tag{25}$$

where f_{ABC} are Lie algebra structure functions analogs (22).

To obtain a correct theory of discretised membranes we must impose the residual constraint (8) of the membrane, this implies a set of constraints $\{L_A\}$ is a realization of the 2 dimensional diffeomorphisms algebra that preserves areas over our discretised membrane model

$$T = \{p_I, x^I\}_{\mathcal{LB}} = (x^{IA} p^B_I) f^C_{AB} Y_C = 0 \tag{26}$$
$$\Rightarrow L_A = f_{ABC} x^{IB} p^C_I = 0. \tag{27}$$

that are the 2-brane analogous of Virasoro constraints

We may define a first quantization theory for the discretised membrane, where the Hilbert space consists of the scalar wave functions valuated over the infinite set of coefficients $x^I_A(\tau)$ instead of $x^I_A(\tau, \sigma^1, \sigma^2)$ with $A \in \mathbb{Z}^2$, $I = 1, ..., d-2$ where $A = (0, 0)$ corresponds to $x^I_{(0,0)} = x^I_o(\tau)$ and $p^I_{(0,0)} = p^I_o(\tau)$.

$$\Phi(x^{IA}) : \mathbb{R}^{N(d-2)} \to \mathbb{C} \tag{28}$$

The operators position and momentum are defined in the Schrödinger representation as

$$X^{IA}|\Phi>= x^{IA}|\Phi> \quad \text{and} \quad P_{IA}|\Phi>= -i\frac{\partial}{\partial x^{IA}}|\Phi> . \tag{29}$$

Eliminating the zero mode from (25) we get the Schrödinger equation

$$\left[-\frac{1}{2}(\frac{\partial^2}{\partial x_0^{IA^2}} + (\frac{1}{2}f_{ABC}x^{IB}x^{JC})^2\right]|\Phi>= E|\Phi> . \tag{30}$$

that jointly with the residual constraints L_A are the equations for the wave functions.

$$-if_{ABC}x^{IA}(\frac{\partial}{\partial x^{IA}})|\Phi>= 0 \tag{31}$$

It is easy to see that coefficients L_A of the constraint T satisfy the same algebra with Poisson Brackets than the basis Y_A(20) in term of Lie Bracket. In fact

$$\{L_A, L_D\}_{\mathcal{PB}} = (f_{ABC}f_{DCF} - f_{DBC}f_{ACF})x^B p^F$$

but $T_A = f_{A(BC)}$ correspond to the matrix adjunct representation, that satisfy the commutator Algebra

$$[T_A, T_D] = f_{A(BC)}f_{D(CF)} - f_{D(BC)}f_{A(CF)} = f_{ADE}f_{E(BF)} \tag{32}$$

$$\text{and} \quad [T_A, T_D] = f_{ADE}T_E \text{ then } \quad \{L_A, L_D\}_{\mathcal{PB}} = f_{ADE}L_E. \tag{33}$$

This theory is not uniquely defined, for example consider a closed membrane with periodic boundary conditions $x^{IA}(0,\sigma^2) = x^{IA}(2\pi,\sigma^2)$ and $x^{IA}(\sigma^1,0) = x(\sigma^1,2\pi)$. That is the topology of a Torus, then winding number, charge from wrapped coordinates and modular invariance must be considered as in [17]. But for the open bosonic membrane, neither of these needs to be considered and the Hamiltonian is just (25) with constraints (27). As there is still a residual gauge invariance generated by the constraints we will proceed to the BRST analysis.

Effective BRST Hamiltonian for the discretised open membrane

The BRST generating charge for a closed constraint algebra [25] is

$$\Omega = c^A L_A - \frac{1}{2} c^A c^B f^C_{AB} \mu_C. \tag{34}$$

Following a modified BFV [19] approach The Functional Integral including the BRST invariant terms is given as:

$$I = \int Dz \; e^{i \int dt \; p\dot{x} - \mu_A \dot{c}^A - H_{o(brst)} - \hat{\delta}(\lambda^A \mu_A) + \hat{\delta}(\underline{c}_A \chi^A)} \tag{35}$$
$$Dz = Dp \, Dx \, Dc \, D\mu \, D\underline{c} \, DB \, D\Theta$$

where $\hat{\delta}$ is the BRST transformation, p, x and μ, c are canonical variables, while the others are auxiliary variables not canonical.

The BRST invariant Hamiltonian is given by

$$\mathcal{H}_{o(brst)} = \mathcal{H}_o + \mu_a \, {}^{(1)}V^A_B \, c^B \tag{36}$$

where the structure functions ${}^{(1)}{V_B}^A$ are the coefficients of the Poisson brackets

$$\{L_B(\sigma), \mathcal{H}_o(\underline{\sigma})\}_{\mathcal{PB}} = {}^{(1)}V^A_B \, L_A. \tag{37}$$

We obtain that this coefficients are null in virtue of

$$\{Y^F L_F, \mathcal{H}_o\}_{\mathcal{PB}} = \{T, \mathcal{H}_o\}_{\mathcal{PB}} = \epsilon^{ij} \partial_i \{\phi_j(\sigma), \phi_3(\underline{\sigma})\}_{\mathcal{PB}} = 0 \tag{38}$$

this implies that ${}^{(1)}{V_B}^A = 0 \;\; \forall \quad A$ and B.

So we get the invariant BRST Hamiltonian

$$\mathcal{H}_{o(brst)} = \mathcal{H}_o = \frac{1}{2} \, p^{IA}.p_{IA} + \frac{1}{4} \, f^C_{AB} \, f_{CED} \, x^{IA} x^{JB} x^{IE} x^{JD}. \tag{39}$$

The transformation laws of a object depending on canonical variables are given by Poisson brackets

$$\hat{\delta} F(p, x, \mu, c) = -1^{\epsilon_F} \{F, \Omega\}_{\mathcal{PB}} \tag{40}$$

while the non canonical variables the transformation laws are given, in modified BFV approach, by auxiliary fields B_A and Θ^A such that

$$\begin{aligned} \hat{\delta} \underline{c}_a &= B_A & \qquad \hat{\delta} B_A &= 0 \\ \hat{\delta} \lambda^A &= \Theta^A & \qquad \hat{\delta} \Theta^A &= 0 \end{aligned} \tag{41}$$

Using the transformation laws (40) for $\hat{\delta} \mu_A$ and (41) for $\hat{\delta} \lambda^A$ and $\hat{\delta} \underline{c}_a$ into (35) we get

$$I = \int Dz \; e^{i\int dt\; p\dot{x} - \mu_A \dot{c}^A - \mathcal{H}_{o(brst)} - \lambda^A (L_A - c^B f^C_{AB}\mu_C) - \Theta^A \mu_A + B_A \chi^A - \underline{c}_A \hat{\delta}\chi^A} \tag{42}$$
$$Dz = Dp\, Dx\, Dc\, D\mu\, D\underline{c}\, D\lambda\, DB\, D\Theta \quad ,$$

note that $\mathcal{H}_{o(brst)}$ is the BRST invariant Hamiltonian (39), and $Dx = \prod\limits_{I,A} dx^{IA}$

Note that $\hat{\delta}\mu_A$ gives BRST extended constraints using the BRST charge Ω (39)

$$L_{A(brst)} = L_A - c^B f^C_{AB}\, \mu_C \tag{43}$$

that generates the same constraint algebra (33) than L_A

$$\{L_{A(brst)}, L_{D(brst)}\}_{\mathcal{PB}} = f_{ADF}\, L_{F(brst)} \; . \tag{44}$$

We will now fix into (42) the residual gauge freedom generated by the constraints taking

$$\chi^C = \lambda^C - \kappa^C \tag{45}$$

where κ^C is a suitable collection of constants, that is, independent of (σ_1, σ_2).

Integrating (42) in the auxiliary variables $D\Theta^A$, from the linear terms in Θ^A we obtain products of dirac deltas $\delta(\underline{c}_A - \mu_A)$ that eliminates the integrals in $D\underline{c}_A$

$$I = \int Dz \; e^{i\int dt\; p\dot{x} - \mu_A \dot{c}^A - \mathcal{H}_{o(brst)} - \lambda^A (L_A - c^B f^C_{AB}\mu_C) + B_A(\lambda^A - \kappa^A)} \tag{46}$$
$$Dz = Dp\, Dx\;\, Dc D\mu\, DB\, D\lambda$$

the linear terms in the action gives dirac deltas $\delta(\lambda^A - \kappa^A)$ on the gauge fixing when integrated in DB, then integrating on $D\lambda$ the gauge is evaluated

$$I = \int Dz \; e^{i\int dt\; p\dot{x} - \mu_A \dot{c}^A - \mathcal{H}_{o(brst)} - \kappa^A (L_A + \mu_C \;\, f^C_{AB} \;\, c^B)} \tag{47}$$
$$Dz = Dp\, Dx\;\, Dc D\mu$$

Finally integrating on the $D\mu$, as the antighost appears linearly, we get a dirac delta on the equations of motion for the Fadeev-Popov c^A

$$I = \int Dz \prod_A \delta(\kappa^C \;\, f^A_{CB}\;\, c^B - \dot{c}^A)\; e^{i\int dt\; p\dot{x} - \mathcal{H}_{o(brst)} - \kappa^A L_A} \tag{48}$$
$$Dz = Dp\, Dx\, Dc$$

the gauge fixing conditions (45) allows the simplification $\kappa^C = 0$ for $\dot{c}^A = 0$, that is the case of non time dependence of the ghosts fields. Then ghosts fields $c^A = c^A(\sigma_1, \sigma_2)$ and also $\mu_A = \mu_A(\sigma_1, \sigma_2)$ are stationary, and the last term right at action is null. So the only not null terms in the exponential defines the effective Hamiltonian

$$\mathcal{H}_{eff} = \mathcal{H}_{o(brst)} = \frac{1}{2}\, p^{IA}.p_{IA} + \frac{1}{4}\, f^C_{AB}\, f_{CED} x^{IA} x^{JB} x^{IE} x^{JD} \tag{49}$$

that only in this particular gauge choice coincides with the BRST invariant Hamiltonian. Note that this is implies an important simplification and decouples from the ghost modes.

Discreteness spectrum of the open membrane

In this section we will prove that the spectrum of the membranes is discrete. This proof is performed directly on the discretised open membrane model taking into account the local constraints L_A and the residual BRST invariance in a particular gauge fixing choice. The BFV theorem [15] establishes that: *all observables are independent of constraints representation and gauge fixing conditions,* so it remains valid in any other gauge and our proof will not depends in the limit $N \to \infty$ of the $SU(N)$ Yang Mills models [8].

We will use a corollary due to B. Simon [26] of a beautiful theorem due to Fefferman and Phong [27] about the spectral dimension of the quantum Hamiltonian. This Corollary establishes that the number of eigenvalues (counting multiplicities) of the Hamiltonian is finite for every finite total system energy value, if the Hamiltonian operator for the quantum system is

$$\mathcal{H} = -\nabla^2 + V(X), \quad x \in \mathbb{R}^m \quad \text{and} \quad V(x) \geq 0 \tag{50}$$

where the potential $V(x)$ can be written as a sum of homogeneous polynomials of degree 2

$$V(x) = \sum_{j=1,..,m} Q_j^2, \text{ that satisfies } \sum_{j=1,..,m\,;\,\alpha=1,..,n} \left(\frac{\partial Q_j}{\partial x^\alpha}\right)^2 > 0 \;\forall\; x^\alpha \neq 0 \tag{51}$$

So we are going to prove that the effective BRST Hamiltonian accomplishes all these conditions.

From (49) it is evident that the potential could be written as

$$V(x) = \sum_{C,I,J} (Q^C_{IJ})^2 \; , \tag{52}$$

It is evident that $V(x) \geq 0$, where Q_j is replaced with $Q^C_{IJ} = f^C_{AB} x^A_I x^B_J$ are homogeneous polynomials of degree 2 of x^B_J.

We must calculate the left hand side of (51) that is obtained from (39) by x^B_J and squaring

$$\sum_{A,E,I} x^{IA} \, f_{ABC} \, f_{CEB} \, x^E_I = (x^{IA} T_A, x^E_I T_E) \tag{53}$$

as $T_A = f_{A(BC)}$ correspond to the adjunct representation (33) and $x^{IA} f_{A(BC)}$ and $f_{E(BC)} x^E_I$ are forms valued on the adjoint representation then the product in (53) correspond to the usual definition of the Killing product.

$$(x^I, x_I)_K = (x^{IA} T_A, x^E_I T_E) = x^{IA} x^E_I K_{AE} \tag{54}$$

We only have to prove that this Killing product is not negative for all $x^I \neq 0$.

The Killing metric is diagonal

$$K_{AE} = T_A T_E = (B \times C)(B \times C) \delta^{B+C}_A \; \delta^{B+C}_E$$

Taking the trace, we obtain

$$K = tr \; (T_A T_E) = \sum_B \sum_C (C \times B)^2 > 0 \quad \text{such that } A = B + C = E \tag{55}$$

then the Killing product (54) is positive definite when $x^A_I \neq 0$

$$(x^I, x_I)_K = K \; \eta^{IJ} \; x^A_I x^A_J = K \left\| x^A_I \right\|^2 > 0 \tag{56}$$

because the light cone transverse metric η^{IJ} is diagonal and positive so $(x^I, x^I)_K$ is a sum of positive terms.

This discrete membrane model accomplishes all the above conditions then we conclude that the spectrum is discrete for every finite amount of energy.

This proof is similar to that of [28] are some differences to be considered, here the gauge group is not Yang Mills with gauge group $SU(N)$ neither for N finite nor infinity. The field is expanded in Fourier modes, and the result could change, if a different basis or a different topology [24] of the base space I_2 were chosen.

This result could be used to explain apparent contradictory results for the membrane spectrum. The matrix representation of $SU(N)$ is indeed a non faithful realization of the complete residual symmetry of the discretised membrane, generated by $\{L_A\}$, so the 2-dimensional diffeomorphisms is a larger group than $SU(N)$ even in the $N \to \infty$ limit. The N^2-1 generators of the $N \times N$ matrix representation of $SU(N)$ are [23][24]

$$J_A = J_{mn} = \frac{i}{2} N\ e^{mn\pi i/N} g^m h^n \quad \text{with} \quad 0 \leq m, n \in \mathbb{Z} < N \quad \text{and} \quad A \neq (0,0) \tag{57}$$

where $g^N = I_{N\times N}$ and $h^N = I_{N\times N}$ are unitary, traceless matrices, roots of the identity in N dimensions, such that $h.g = e^{2\pi i/N} g.h$ with commutation algebra

$$[J_A, J_B] = \frac{N}{\pi} \sin(\frac{\pi}{N} A \times B)\ J_{A+B} \tag{58}$$

that in the limit $N \to \infty$ almost reproduces (33), but the values of the indexes of J_{mn} takes only non negatives values $(m, n \geq 0)$ while the indexes of $T_A = T_{mn}$ and $L_A = L_{mn}$ takes both positive and negative values ($A \in \mathbb{Z}^2 - (0,0)$), because completeness of Fourier basis needs both positive and negative frecuencies. So the number of generators is larger than in the $SU(N)$ case, implying a larger symmetry even in the $N \to \infty$ limit.

If we considers only non negative values for m, n, that is restricting the discretised membrane symmetry to a subgroup that could be realized by $SU(N \to \infty)$, then the theorem fails because conditions are not accomplishes and (55) is to be null for $x_I^A \neq 0$. For example: take $A = (0, n)$, then it is required that $B = (0, n_1)$ and $C = (0, n_2)$ such that $n_1 + n_2 = n$ with $n_1, n_2, n \geq 0$. So $C \times B = 0$ and (56) is null for $\left\| x_I^A \right\|^2 > 0$

$$(x^I, x_I)_K = \sum_{B=(0,n_1)} \sum_{C=(0,n_2)} (0.n_2 - 0.n_1)^2 \left\| x_I^A \right\|^2 = 0 \tag{59}$$

then the spectrum is continuous with abelian valleys or string like directions $(0, n)$ consistent with the results of [8][4][9].

As we considers the complete symmetry, with positive and negative values for m, n. This implies more generators and also more gauge fixings, that allows to eliminate the string like spikes. Indeed, for $A = (0, n)$, we may take $B = (m, n_1)$ and $C = (-m, n_2)$ such that $n_1 + n_2 = n$ then

$$K = tr\ (T_A T_E) = \sum_B \sum_C (m.n_2 + m.n_1)^2 > 0 \tag{60}$$

so for the complete 2D diffeomorphisms symmetry group there will be a discrete spectrum. The same will happens in compactified membranes on the torus, for which the Fourier Series are also complete (but with uniform convergence) in concordance with [6][7][17]

Conclusions

In this paper we obtain the effective Hamiltonian for the open discretised membrane, starting from a partial gauge fixing, namely the conformal light cone that allows a residual gauge: 2D area preserving diffeomorphisms. This residual gauge, forms a first rank algebra in the sense of [15] for which the BRST Functional Integral was obtained (42) following a modified BFV approach [19], including Fadeev-Popov ghost, gauge fixings and ${}^{(1)}V_B{}^A = 0$ terms that allows a complete gauge fixing $\lambda^A = \kappa^A$. The ghost dependent terms could be integrated producing a delta-measure (46) of the functional integral, so the BRST effective Hamiltonian includes only bosonic terms (49) that resembles a classical Yang Mills but with the symmetry algebra that could be represented unfaithfully as $SU(N)$ algebra in the $N \to \infty$ limit, but is not $SU(N)$ because it has more generators than Yang Mills theory.

Finally we conclude that the effective Hamiltonian for the open discretised membrane, accomplishes the conditions of the Simon [26] and Fefferman and Phong theorems [27], this means that for finite energy the spectrum of the membrane is discrete and with not abelian valleys for the potential, so string like instabilities will not arise. Although this result was obtained in a particular gauge fixing, due to the BFV theorem [25], the result must remain valid in all gauges.

Apparent contradictory results about the discreteness- continuity of the membrane spectrum could be explained by the fact that $SU(N)$ is an unfaithful realization of the residual symmetry and not the complete diffeomorphisms group. It is not known if this result could be equally extended to the supermembrane.

In Physical terms, open discretised membrane is an example of a classical system of particles with a not a confining potential but that is a quantum mechanical confining system for the wave functions of that particles, that are the Fourier modes of the bosonic membrane. The way in which Supersymmetry breaks or not, the stability of the spectrum for the open Supermembrane case must be carefully studied to include the complete gauge fixings of the complete residual symmetry, that as shown here, is larger than Yang Mills $SU(N \to \infty)$ models.

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