

The Concept of Risk in Portfolio Theory

El Concepto de Riesgo en Teoría de Portafolios

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ABSTRACT. This article shows that the systematic risk of an asset depends on two factors: it is proportional to the standard deviation of its rate of return and to its correlation with the optimal portfolio into which the asset is included. The theory is stated as a relation between mean and standard deviation, not between mean and variance. The Beta factor is explained as a measure of systematic risk in portfolio theory, independent of any asset valuation model. Risk, measured by Beta, is the contribution of the asset to the risk of the efficient portfolio into which it is included. This definition is verified by demonstrating that the standard deviation of the portfolio, that is, the portfolio's risk, can be calculated as the weighted sum of the individual, marginal risks of the assets, by coincidence with the conditions of Euler's theorem. The CAPM proposal is explained and criticized, with emphasis in the assumption about a unique market portfolio. The alternative concept of a straight line frontier with many different optimal portfolios, proposed by Sharpe in his seminal article, is discussed. The concepts developed in this article should be helpful for future studies of risk and its effect on the valuation of financial assets, given the lack of empirical evidence found for the CAPM model after 40 years of research.

Keywords: Portfolio, diversification, systematic risk.

RESUMEN. En este artículo se muestra que el riesgo sistemático de un activo depende de dos factores: es proporcional a la desviación estándar de su tasa de retorno y a su correlación con el portafolio óptimo dentro del cual está inserto. La teoría se plantea como una relación entre media y desviación estándar, no como media – varianza. Se explica el factor Beta como una medida de riesgo sistemático, perteneciente a la teoría básica de portafolios, e independiente de cualquier modelo de valoración de activos. El riesgo, medido por Beta, es la contribución del activo al riesgo del portafolio eficiente dentro del cual está inserto. Esto se verifica al demostrar que la desviación estándar del portafolio, que es su riesgo, es igual a la suma ponderada de los riesgos individuales o marginales de los activos, por coincidencia con las condiciones del teorema de Euler. Se explica y critica la propuesta del CAPM y en particular el supuesto de un portafolio único de mercado, analizando el concepto alternativo de frontera rectilínea con muchos portafolios óptimos planteado por Sharpe en su artículo seminal. Los conceptos planteados deben ser útiles para futuros estudios del riesgo y su efecto sobre la valoración de activos financieros, dada la falta de respaldo empírico para el modelo CAPM después de más de 40 años de investigaciones.

Palabras clave: Portafolio, diversificación, riesgo sistemático.

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INTRODUCTION

Asset Valuation Theory is based on the concept of portfolio developed by Markovitz (1959). Investors do not value each asset independently but in relation to the set or portfolio of assets into which the asset is inserted. The return of an asset can be measured independently of the other assets. The risk of the asset, however, is not measured by the standard deviation of the asset's rate of return, but as the contribution of the asset to the standard deviation of the portfolio. This contribution depends on the correlation of the asset with the returns of the other assets that make up the portfolio. The purpose of this article is to discuss the concept of financial risk, the way it must be measured, and its possible use in valuation models such as the CAPM. Some basic, well known principles of portfolio theory are discussed to explain the concept of risk and its applicability, mainly to underline the aspects that determine the proposed approach.

FINANCIAL RISK

The word "risk" is used in financial theory to describe to different kinds of probabilistic phenomena, namely, catastrophes and variability. The first is the usual meaning, referring to the possibility of the occurrence of a catastrophic event such as an accident, the failure of a business firm or an unexpected fall in the price of shares. The opposite of a catastrophe is a lucky event such as winning a lottery, the discovery of a wonderful business opportunity or an unexpected rise in the price of shares. Statistically, these are binomial events, that may happen or not with determined non-symmetrical probabilities. There is no generic word to designate these events except "luck", which can be good or bad. The risk of catastrophe is managed with prevention and with insurance: efforts are made to prevent a fire, and insurance can be taken against the possibility of the fire happening in spite of the undertaken precautions. Insurance does not prevent the catastrophe but protects against the possible financial consequences. Insurance can be taken against the financial consequences of not having made an investment that is ex-post profitable. The financial market has an extraordinary insurance mechanism which is the derivatives market. In addition, strict prevention mechanisms are employed, frequently expressed in terms of ratios, used as limits or as warning signals, which are an essential part of corporate government and of regulatory systems.

The second kind of risk, frequently called "financial risk", is the variability of the rate of return of investments. Because the distributions of the rates of returns are assumed to be normal, variability is measured by the standard deviation. The tool for managing symmetric variability risk is portfolio theory, the main subject of this article.

As to the integration of both kinds of risk into a unified theory, there is very advanced work already done but it is far from complete. Usually, the problem of asset valuation – which includes the problem of the financing of the firm – is studied on one side. And as a separate subject, comes the study of options theory, including real options and the valuation of forward contracts. The main efforts towards the integration of both aspects are done in the context of project evaluation.

THE CONCEPT OF PORTFOLIO

The concept of portfolio derives from the fact that when several investments are combined – rather than putting all the eggs in the same basket – it happens that they do not all move exactly in the same way: in any given period some of them have high returns; others have lower or negative returns. Is it impossible to guess which ones will have high returns to invest only in those assets. If investment is made in a portfolio of several assets it can be reasonably assumed that not all of them will have negative results: but it will not happen either that they all have excellent returns. Results are compensated among them, eliminating the risk of very low returns at the cost of also eliminating the possibility of having very high returns for the portfolio as a whole.

Rigorously defined, diversification consists in obtaining a standard deviation for the portfolio that is lower than the weighted average of the standard deviation of the component assets. This happens because the correlation among the different assets is not perfect. The compensation effect produced when forming a portfolio is more than a simple averaging. Formally: (Markovitz, 1959). The fact that the weights add up to one means that ten whole of the available recourses is invested in the portfolio. Any cash not invested in papers is itself an asset of the portfolio, with zero return and zero risk if there is no inflation.

The concept of portfolio is derived from two crucial assumptions which are risk aversion and normally distributed rates of return.

$$P = w_1 A + w_2 B + \dots + w_N N \quad (\sum w_i = 1)$$

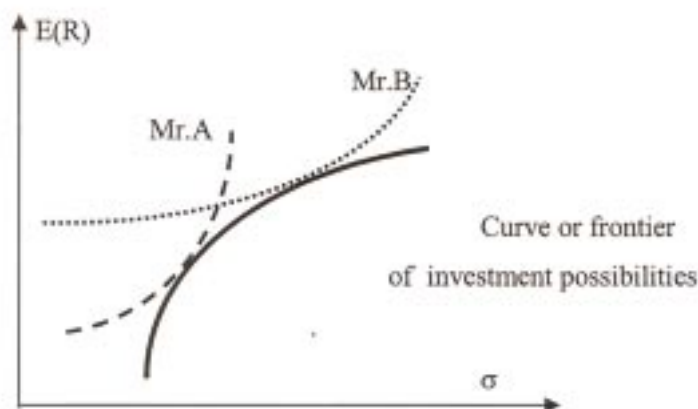
$$\sigma_P < \sum w_i \sigma_i \quad (\text{in the general case}) \quad (\text{see Annex 1})$$

RISK AVERSION

Financial theory assumes that all investors are risk averse, so that stability is a desirable objective sought after by everybody. The degree of risk aversion can be higher or lower, but there are no investors for whom risk is a desirable thing. This is not at odds with the idea that the essential entrepreneurial attitude is that of bearing risk, which leads to say that entrepreneurs "love risk": what a businessman loves is the return associated to a certain risk; he can assume large risk in exchange for a large expected return; he loves the

opportunity coming to him because he takes risk when other people do not dare to do so.

Risk aversion is shown by the shape and position of the indifference curves between risk and return. If the investor is risk averse, his curves are positively sloped (contrary to normal indifference curves) and they are convex. One person is more averse than other if, given the same investment possibilities, he prefers a portfolio with lower expected standard deviation, even if it offers a lower expected return. In the graph that follows, Mr. A is more averse than Mr. B. In more rigorous terms, a person is risk averse if his or her utility function of wealth is quadratic, so that he or she prefers a known sum to a game with the same expected value. This utility function produces the indifference curves that were explained. .



NORMALLY DISTRIBUTED RATES OF RETURN

The second critical assumption of finance theory is that the rates of return of traded assets are normally distributed. Eugene Fama showed that the normality assumption is adequate, even if actual distributions are stable paretian, with higher probabilities for values that are far from the mean than those shown by the normal distribution. (Fama, 1972)

The normal distribution is fully described by only two parameters, the mean (μ) and the standard deviation (σ). If a variable x is normally distributed:

$$X \sim N(\mu, \sigma)$$

ts probability density function is:

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

In the formula, x is the value for which we want the probability; μ is the population mean and σ is the standard deviation. The formula appears to be complicated, but it contains only the mathematical constants e and π ; and the parameters μ y σ . If the parameters are known the whole curve can be drawn. A single distribution table, known as the standard nor-

mal distribution, can be used to compute any normal probability, because:

$$(x-\mu)/\sigma \sim N(0, 1)$$

Because investments have two magnitudes, namely, return and risk; and as a normal distribution is completely described by two parameters that are E and σ , in portfolio theory the expected value of the rate of return is called "return" and the standard deviation of the rate of return is called "risk". It is clear then that portfolio theory deals only with symmetric

variability risk measured by standard deviation.

A portfolio is a sum of random variables that are the rates of return of the component assets. The expected value of the rate of return of a portfolio, $E(R_p)$, is the (weighted) sum of the expected values:

$$E(R_p) = \sum w_i E(R_i)$$

The standard deviation of the rate of return of a portfolio is the square root of its variance. And the variance of a sum of random variables is the weighted

average of the covariance among all the components. The covariance matrix must be formed and all the terms must be summed up:

Weights: ($\sum w_i = 1$)	w_1	w_2	w_3	w_n
w_1	σ_{11}	σ_{12}	σ_{13}	σ_{1n}
w_2	σ_{21}	σ_{22}			.
w_3	σ_{31}	σ_{32}	σ_{33}		.
	.			.	.
	.			.	.
	.			.	.
	.			.	.
	.			.	.
w_n	σ_{n1}			σ_{nn}

The values along the diagonal are the covariances of each asset with itself, that is, their variances. The

rest of the terms are covariances. The triangle under the diagonal is identical to the triangle over it, because:

$$\sigma_{xy} = \sigma_{yx}$$

$$\sigma_p^2 = \sum w_i w_j \sigma_{ij}$$

$$= \sum w_i^2 \sigma_i^2 + 2 \sum w_i w_j \sigma_{ij} \quad (\text{for } j > i)$$

In this square, n by n , matrix, there are n^2 terms, of which $n^2 - n$ are variances and all the rest are covariances. I.e., with 20 assets there would be 20 variances and 380 covariances. It can be seen that portfolio variance depends mostly on covariances; but

this does not mean that standard deviations are not important, because higher σ means higher covariance.

Because of the importance of covariance, the risk of an individual asset is not the standard deviation of its

rate of return, but, the contribution that the asset makes to the standard deviation of the rate of return of the portfolio: a high risk asset is one that causes an increase in σ_p ; a low risk asset is one that reduces σ_p . In mathematical terms (Fama - Miller, 1972):

$$\text{Risk of an individual asset "A"} = \delta \sigma_p / \delta a$$

where "a" is w_A , the weight of asset A in the portfolio. This partial derivative looks menacing for a person with little mathematical ability; but the solution turns out to be simply:

$$\delta \sigma_p / \delta a = \sigma_{AP} / \sigma_p$$

The variability risk of an asset is measured by its covariance with the portfolio over the standard deviation of the portfolio. It happens, then, that risk can be measured by the simple linear regression coefficient Beta, because Beta and risk are defined in a similar way.

$$E(R_A) = \alpha + \beta E(R_p) + \varepsilon$$

The Beta factor in the model is defined as:

$$\begin{aligned} \beta &= \sigma_{AP} / \sigma_p^2 = (\sigma_{AP} / \sigma_p) / \sigma_p \\ &= \text{Risk of the asset} / \text{risk of the portfolio} \end{aligned}$$

Beta measures the asset's risk **as a proportion of** portfolio risk. In fact, Betas take values around one: a Beta value higher than one means that the asset's risk is higher than the risk of the portfolio; if the asset is included into the portfolio, or if the proportion invested in the asset is increased, the portfolio risk measured by its standard deviation increases. Contrary wise, a Beta lower than one means that the asset decreases the risk of the portfolio. Beta is usually called "the risk" of the asset, a habit that can lead to conceptual mistakes.

The fact that Beta - used in the most universal and better known statistical model - has a definition that is similar to that of financial risk is a fortunate circumstance, logical and self evident once researchers discovered it. It is a statistical fact that does not need to be deduced or derived from any asset valuation model: Beta must be used to measure variability risk if returns are normally distributed and

investors are risk averse.

If asset risk is not measured by its standard deviation, it may not be clear how it can be higher or lower than the portfolio standard deviation. For this purpose, it is convenient to state covariance, the critical component of portfolio variability, in terms of correlation coefficients:

$$\rho_{AP} = \sigma_{AP} / \sigma_A \sigma_p$$

$$-1 \leq \rho \leq 1$$

The correlation coefficient between two variables (ρ_{AP}) is the covariance over the standard deviations. Covariance can take values from minus infinity to plus infinity, making it very hard to understand. The correlation coefficient, instead, is a standardized covariance that ranges only between minus one and plus one. A value of one means that movements in one variable allow perfect prediction of the movement of the other variable. From the previous definition:

$$\sigma_{AP} = \rho_{AP} \sigma_A \sigma_p$$

Covariance among two variables is "Rho Sigma Sigma", the correlation coefficient times the product of the standard deviations. It can be said thus:

$$\begin{aligned} \text{Risk of A} = \delta \sigma_p / \delta a &= \sigma_{AP} / \sigma_p \\ &= \rho_{AP} \sigma_A \sigma_p / \sigma_p \\ &= \rho_{AP} \sigma_A \end{aligned}$$

This is a crucial result that shows that the relevant risk of an asset is a fraction of its standard deviation, the fraction being ρ_{AP} , the correlation coefficient among the returns of the asset and those of the portfolio into which the asset is included. ρ_{AP} is practically always lower than one, so that the risk of an asset included into a portfolio is lower than its standard deviation. When ρ_{AP} is negative, the asset's risk is also negative, so that in equilibrium the expected return can be negative: investors might be willing to accept a negative return in exchange for a large reduction in portfolio risk. It can be assimilated to variability insurance, for which the consumer would be willing to pay.

Systematic and un-systematic risk

Total risk of the asset is its standard deviation (σ_A). But risk is reduced when the asset is included into a portfolio, leaving only the fraction of risk that is called “systematic”, that cannot be diversified away because it is caused by factors that affect the whole “system”, or every asset in the system. The fraction of risk that is eliminated is called non systematic, idiosyncratic, or diversifiable risk. (See Annex 1)

Total risk of the asset = σ_A

Relevant, non diversifiable, or systematic, risk = $\rho_{AP} \sigma_A$

Non – systematic, diversifiable risk = $(1 - \rho_{AP}) \sigma_A$

As can be seen, the systematic risk of an asset is directly proportional to its standard deviation. But risk of the asset is only a fraction (ρ_{AP}) of σ_A . Covariance is proportional to the factor ($\rho_{AP} \sigma_A$) so that it also measures risk, but it does not offer information about the composition of risk, how much of it is due to σ_A and how much is due to ρ_{AP} .

In equilibrium, the return of an asset depends only on its systematic risk. Non – systematic risk “does not pay”, is not rewarded by a higher expected return because the investor can diversify it away. Idiosyncratic risk becomes zero when ρ_{AP} is equal to one: in this case diversification is ineffective because all the risk is systematic. It must be noted that idiosyncratic risk is not the same as what in regression analysis might be called “unexplained standard deviation”, that would be the square root of unexplained variance: $(1 - R^2) \sigma_A^2$. Explained and unexplained variances add up to σ_A^2 ; but the equality does not hold when square roots are taken.

Put this way, it can be readily seen that Beta is a measure of risk, equal to asset risk over portfolio risk.

$$\beta = (\sigma_{AP} / \sigma_P) / \sigma_P = \rho_{AP} \sigma_A / \sigma_P$$

Having explained covariance in terms of ρ_{AP} , the risk of the portfolio can be computed as the (weighted) sum, or average, of the relevant risks of the assets that make up the portfolio. Each line of the covariance

matrix is the weighted sum of the covariances of one asset with all the other assets: this is equal to the covariance of the asset with the portfolio. If the lines of the matrix are added up, the variance of the portfolio can be expressed as follows:

$$\sigma_P^2 = \sum w_i \sigma_{iP}$$

And if covariances are expressed in terms of correlation coefficients we have:

$$\sigma_P^2 = \sum w_i \rho_{iP} \sigma_i \sigma_P$$

Dividing both sides by σ_P yields:

$$\sigma_P = \sum w_i \rho_{iP} \sigma_i$$

The last formula states that the risk of a portfolio, which is its standard deviation, is the weighted sum of the individual risks of the component assets. Seen this way, it becomes evident that a risk with a value higher than the average, translated into a Beta higher than one, will increase the average; and a lower value will reduce the average.

This formula is seldom used in literature because it does not have general application in other statistical problems. It does not strike as obvious that the sum of the marginal risks be equal to total risk because, foremost functions, it is not true that the sum of the marginal values be equal to the total value. What happens here is that the a portfolio is a particular case of Euler's theorem, which states that if a production function is homogeneous of degree one, the sum of the marginal products is equal to total production. It implies constant returns to scale. This applies to the portfolio function because a small portfolio and another, larger one, with the same composition behave exactly alike. Anyway, the easiest way to become convinced of this fact is to build a small portfolio, then compute σ_A and ρ_{AP} for every asset, add them up and verify that the sum of the marginal risks is equal to the standard deviation of the portfolio.

Standard deviation cannot be negative, by definition. But the variability of a portfolio can be reduced to zero

if an adequate proportion of assets with negative risk is included. If the proportion is increased beyond this point, the resulting portfolios have positive σ_P but their correlation with efficient portfolios will be negative, as it is the case in the lower, convex, inefficient section of the minimum- σ frontier.

Betas are proportional to the risk of each asset and they can also be averaged. For example, if a firm invests in several fields, the firm's Beta is the average of the Betas of the different fields; and it also equal to the average of the Betas of the firm's liabilities. The average of all the assets in an optimal portfolio is one, if the Betas are measured against that portfolio, not against a proxy that might have higher or lower risk.

The last clarification is necessary, because the optimal portfolio of a person might be comprised by several "sub-portfolios". The risk of an asset is its contribution to the complete optimal portfolio. Each "sub-portfolio" is an asset that has a Beta equal to the weighted average of the Betas of the component assets. In other words, it has a systematic risk that is equal to the weighted average of the systematic risks of the assets.

Everything said this far is logical-mathematical analysis based on the assumptions of normality and risk aversion. No mention has been made of any valuation theory. Beta is a portfolio measure that does not need to be derived from a financial model.

THE BETA FACTOR

Inspection of the formula for the risk of an asset yields a very important deduction: Variability risk can be estimated through the simple linear regression Beta coefficient between the returns of the asset and the returns of the portfolio, because the definition of Beta is similar to the definition of financial risk. The following regression model can be proposed:

$$E(R_A) = \alpha + \beta E(R_P) + \varepsilon$$

In this model, the Beta factor is defined and computed as follows:

$$\beta = \frac{\sigma_{AP}}{\sigma_P^2} = \frac{\sigma_{AP}}{\sigma_P} \cdot \frac{1}{\sigma_P} = \frac{\rho_{AP} \sigma_A}{\sigma_P}$$

That is:

$$\beta = \text{Risk of the asset} / \text{risk of the portfolio}$$

Beta is a measure of correlation and it happens to be identical to the risk of the asset divided by the risk of the portfolio. Beta measures risk **as a proportion of** portfolio risk. In fact, Betas take values around one: a Beta higher than one means that the risk of the asset is higher than the risk of the portfolio, so that when the asset is included, or when the proportion invested in the asset is increased, the risk of the portfolio (σ_P) increases. On the contrary, a Beta which is lower than one means that the asset lowers the variability of the portfolio.

In financial literature is commonly said that Beta is "the risk" of the asset, but this can lead to conceptual mistakes. Even if Beta is usually interpreted as "sensitivity" of one variable with respect to the other, in portfolio theory Beta is used to measure the **contribution** of the individual asset to the risk of the portfolio. Assets do not react to the variability of the portfolio but the other way around: the variability of the portfolio is the consequence of the variability of the component assets.

The significance of being able to employ the Beta factor comes from the fact that financial variables must be estimated from historical data. And estimating the future from past data is very difficult, so much so that it can be said that "it doesn't work". However, statistics has been developed to a point that makes it possible. Econometrics or biometrics have studied what sort of problems can be found when estimates are made from historical data; how can those problems be detected; and what can be done to solve the problems or at least to minimize their effects. In the case of regression, its analysis fills complete books, and the number of recourses available is reflected in the size of the output produced by any good regression software.

That the Beta factor, used in the most universal and well known existing statistical model, have a definition that is similar to financial risk is a fortunate fact, logical and evident once it was discovered by researchers. It is a statistical fact which does not need to be deduced or derived from any asset valuation model: Beta must be used to measure risk if return distributions are normal and investors are risk averse, understanding risk as symmetric variability of the rate of return.

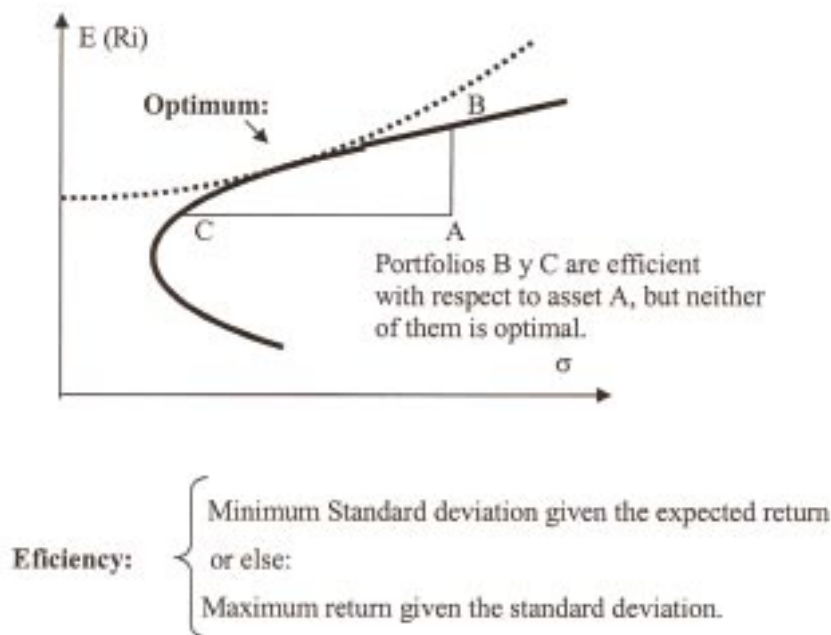
Betas are proportional to the risk of each asset and they, like risk itself, can be averaged, which is a significant operational advantage. For instance, in the case of a firm that invests in different lines of business, the firm Beta is the (weighted) average of the Betas of the different lines; and it is also equal to the average Beta of the firm's liabilities. The average of the Betas of all the assets that form the optimal portfolio of an investor is equal to one, if the Betas are computed against the returns of such optimal portfolio, not against a proxy which might have higher or lower risk than the optimal portfolio.

The last indication is necessary because the optimal portfolio of one individual can be thought of as formed by several "sub portfolios". The risk of an asset must be understood as its contribution to the total portfolio, that is, the optimal portfolio of the investor. Each "sub

portfolio" is an asset that has a Beta equal to the average Beta of the component assets. In other words, it has a systematic risk that is the weighted average of the systematic risks of the assets.

THE EFFICIENT FRONTIER

If investors are risk-averse, optimal portfolios must be "efficient". That is, there must be no other portfolio that offers higher return at the same level of risk, or lower risk at the same level of return. The reason for this is that the investor wants to reach the highest indifference curve available. For any non efficient portfolio, there is at least one that is efficient, located on a higher indifference curve. But it must be noted that the opposite is not true: a portfolio can be efficient but not be optimal. (See following graph)



For a portfolio to be efficient, it must be perfectly diversified: all its non-systematic risk must have been diversified away, so that all of its risk is systematic. Again, the inverse relation is not true: a portfolio can be perfectly diversified but not be efficient, and in that case, it cannot be optimal. The expression "well diversified" is usually employed to mean that a portfolio has many assets. However, a perfectly diversified portfolio might consist of only one asset with zero non-systematic risk; and a portfolio with many assets could have a considerable amount of non-systematic risk.

THE ZERO RISK ASSET: EFFECT OF LEVERAGE

The word "leverage" refers to the increase in expected return that can be obtained by assuming debt. But the most important effect of leverage is the increase in the variability of the rate of return. The word is also used to mean taking up debt, not only the effects of doing so.

The possibility of borrowing or lending at the same rate, assuming that debt will always be paid, can be

represented by the fact that any asset can be combined with asset F, the "risk free" asset. Borrowing represents a negative position - a "short position" - in asset F. Portfolios formed by a risky asset A and the risk-free asset F are described as follows:

$$P = aA + (1-a)F \quad (\text{obviously, } a + 1 - a = 1)$$

$$E(R_P) = aE(R_A) + (1-a)R_F$$

$$\sigma_P^2 = a^2 \sigma_A^2 + b^2 \sigma_F^2 + 2a(1-a) \rho_{AF} \sigma_A \sigma_F$$

The last expression can be simplified because σ_F is zero:

$$\sigma_P^2 = a^2 \sigma_A^2$$

$$\sigma_P = a \sigma_A$$

The factor "a" can be expressed in terms of the Debt/Equity ratio:

$$a = (E+D)/E = 1 + D/E$$

$$\sigma_P = a \sigma_A = (1+D/E) \sigma_A$$

If capital is invested in a "long position" in asset F, factor "a" is less than one and both the expected return and the standard deviation of the portfolio are lower than those of asset A. But when there is debt, investment in F is negative and factor "a" is greater than one: Debt increases expected return and standard deviation in a strictly linear way. Because systematic risk and Beta are proportional to standard deviation, it can be stated that the relation between the Beta of an indebted firm (β_L , for "leveraged") and the Beta of the same firm without debt (β_U , for "unleveraged") is:

$$\beta_L = a \beta_U = (1 + D/E) \beta_U$$

Leverage does not change the correlation of the returns of the firm with the returns of the portfolio. This means that the total return of the assets of a firm and the return on its equity both have the same correlation with the portfolio. The correlation between

the rates of return on equity with leverage and without leverage is equal to one. In fact, if one of the rates is known, the other can be exactly calculated, given the amount of debt and the interest rate paid which are known parameters. If two variables have correlation 1 between them, their correlation with any other variable is the same. This means that leverage of a firm does not change the standard deviation of the optimal portfolio.

$$\rho_{R_L R_L} = 1 \Rightarrow \rho_{R_U R_M} = \rho_{R_L R_M}$$

Use of the previous formula implies full acceptance of the Modigliani – Miller propositions. They state that leverage does not change the value nor the total risk of the firm, as it only changes the way in which return and risk are distributed among shareholders and lenders. Lenders accept a lower return, allowing shareholders to receive a larger portion; but they do not assume risk, meaning that all the risk of the firm must be borne by equity-holders, in the form of a greater variability of their rate of return. In financial terms, this means that the average systematic risk – and Beta – of a firm's assets is equal to the systematic risk – and Beta – of the liabilities, including equity. (Modigliani and Miller, 1959)

If there exists a tax rate "t" that represents a tax shield on debt, the mandatory and certain amount that must be paid for interests is reduced; this means that the risk - and Beta - of the firm and of its equity are reduced, and thus increases their value: (Hamada, 1969)

$$\beta_L = (1 + D/E)(1-t) \beta_U = a(1-t) \beta_U$$

In actual practice, firms usually assume that the Beta of their debt is zero, because it must be paid under any circumstance, no matter what happens in the market. This means that taking debt represents a negative position in asset F, the risk-free asset. The average Beta of the assets (β_A) is usually estimated from the Beta of the equity (β_E) – which in turn is estimated from market prices:

$$\beta_A = (E/(D+E)) \beta_E + (D/(D+E)) \beta_D$$

If β_D is assumed to be zero, the expression comes to the same previous result:

$$\beta_A = (E/(D+E)) \beta_E = (1 + D/E) \beta_E$$

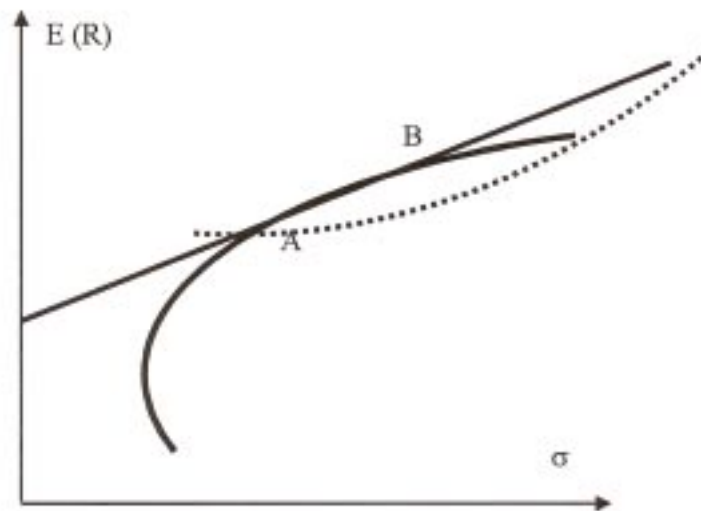
where β_A is β_U and β_E is β_L . The correction for taxes proposed by Hamada must be applied if relevant.

But creditors will not be certain to collect if the firm goes into bankruptcy. The possibility of failure is not at all irrelevant for them. Usually they will not extend credit if the probability of failure is not zero, unless they feel completely secured by collaterals.

THE EFFICIENT FRONTIER INCLUDING ASSET F

We said that the set of risky assets produces the efficient frontier: no risk averse investor will want to

be below the efficient frontier. But if in addition to the risky assets there exists the possibility of investing in a risk free asset "F", there appear portfolios that previously did not exist. Since asset F can be combined with any asset, and since the investor – consumer desires to reach the highest possible indifference curve, it will be advantageous for him to draw a line that starts from F and is tangent to the efficient frontier of risky assets. In other words, he will combine the risk free asset with the tangency portfolio, obtaining a new efficient frontier which is now a straight line. The optimal portfolio **with risk** is the tangency portfolio, but the final optimum portfolio will be the point of tangency of the straight line frontier with an indifference curve. In the graph that follows, the optimum **with risk** is always A, but the final optimum is B, which can be located to the left or to the right of A depending on the individual investor's indifference curves.



The model that is called "Capital Asset Pricing Model" (CAPM) (it is said that the name was originally proposed by Eugene Fama) states that risk measured by Beta is the only factor explaining the differences in expected returns among assets. It is not easy to tell the end of portfolio theory from the beginning of CAPM, because both have been treated as one unity. But CAPM postulates a specific procedure for defining the reference portfolio, which was not analyzed by Markovitz and can be considered as coming after his theory. The expected return that is required from an asset is expressed as follows: (Sharpe, 1964)

$$E(R_A) = R_F + \beta [E(R_P) - R_F]$$

The model states that any asset must offer at least the return offered by a risk-free asset, but in addition it must offer a premium for its variability risk. The risk premium in the market is the difference between the return of the optimal portfolio and the return of the riskless asset. The risk premium for a particular asset is the portfolio premium times the asset beta, which measures systematic risk.

THE ASSUMPTION OF HOMOGENEOUS EXPECTATIONS

For the CAPM to have empirical content it is necessary to surmount an enormous problem: Risk must be measured, in theory with the Beta against a portfolio which is the optimal for each person. If the optimal portfolio is different for each person, risk is also different and there wouldn't be a universally valid way to measure it. For a measure of risk to be useful it must be valid for all investors, or at least for an important group, not for just one of them. To solve this problem, the model uses an assumption of such magnitude that makes it hard to grasp its importance: It is assumed that investors have **homogeneous expectations**, so that every investor in the universe perceives the same investment possibilities, including the same efficient frontier and the same risk free asset, and, consequently, they all come to the same optimal portfolio. If there is one unique portfolio which is demanded, all the assets that have a price must belong to that portfolio, since otherwise they would not be demanded. Following the logic, this unique portfolio could only be the set of all assets, so that all investors should allocate their money among all existing assets – not only those that are traded in public exchanges - and they all should use the same investment proportions. The set of all assets is called M, the “market”. When Betas are computed against “the market” a measure of risk is obtained which, under the assumption of homogeneous expectations, is valid for every investor in the universe. In the previous graph, M must be replaced by P. The portfolio of risky assets in the tangency point of the straight line that comes from F and the efficient risky frontier is “the market”, the set of all existing investment possibilities, including not only shares but all kinds of property and also personal earning capacity. (Fama and Miller, 1972)

Which is, in practice, the market portfolio? Obviously, it is unobservable. The question then becomes: What can we use as a proxy for the market portfolio? The unique portfolio approach has held already 40 years because in the USA it is thought that there exists an adequate empirical counterpart: If it is assumed, for practical purposes, that the universe is USA, that USA is the NYSE – traded firms come from all over the world; produce 60% of USA GNP; and cover all sectors of the economy – and that exchange movements can be measured by well respected indexes, then Standard and Poor's or other similar but theoretically more complete indexes can be used as proxies for “the market”:

$$\text{Universe} = \text{USA} = \text{NYSE} = \text{S\&P}$$

Betas are then computed by means of a lineal regression of the returns of the asset under study against the returns of the chosen index, i.e.:

$$R_A = \alpha + \beta (\text{S\&P}) + \varepsilon$$

Or with an econometrically more robust version of the same equation:

$$R_A - R_F = \beta (\text{S\&P} - R_F) + \varepsilon$$

Years of research have shown that the CAPM, called a “one factor model”, is not strictly valid, that is, that variability is not the only factor explaining the rates of return that are required from each asset. There must be other risk factors that are not included in the estimated standard deviation of the rate of return. For instance, the risk of failure, which means halting the stochastic process of the asset.

The assumption of a unique universal portfolio, equal for all investors, is patently absurd and impossible. The most superficial observation is sufficient to show that, far from being equal, investor's portfolios are all different. But science should not be based on superficial observation: The assumption can be rescued by the possibility that each investor having a “scaled down” version of the global portfolio, which can contain only a small fraction of the set of existing assets, in proportions that are not the same for all investors. For instance, one person could own the whole of one asset, and other person could have none of it. What would make an individual, insignificantly small portfolio a “version” of the global portfolio is that the rates of returns are perfectly correlated. This is only way for the risks of each individual asset to be the same for all investors. The theory assumes, then, that the portfolios of at least the biggest investors in the market, and those of the average investor, have perfect (very high) correlation among them. Since this assumption is not made explicit, the gigantic research efforts done on CAPM have not been used to verify it.

One piece of contradictory evidence with the universal market portfolio is “home bias”, the tendency of investors all over the world to place an abnormally high proportion of their funds in local assets,

presumably to take advantage of information sources and to participate in the firms' governance.

LINEARITY OF BETAS

For any efficient portfolio, betas are strictly lineal: if a Beta is computed for each asset against the efficient portfolio that includes them, and then the betas are shown in a graphic against the expected return of the assets, a strictly lineal relation is obtained. It looks surprising but it is actually quite simple. What any optimization algorithm would do to find an efficient portfolio is to look for the combination of assets that produce the lowest standard deviation for each level of expected return. It would start from any portfolio and would then optimize it improve iteratively. For which assets will it increase the weight, and for which ones will it reduce it? If it finds an asset with a high expected return in relation to its risk measured by Beta, it will increase that asset's weight; and it would decrease the weight of assets with a low relation of return to risk. Each time a weight is changed it has a new reference portfolio and all Betas must be computed again. The process keeps going until the algorithm finds that the returns of each are proportional to their Betas, that is, that they are lineal. At this point, no more reduction of systematic risk is possible for each level of return: the efficient frontier has been reached.

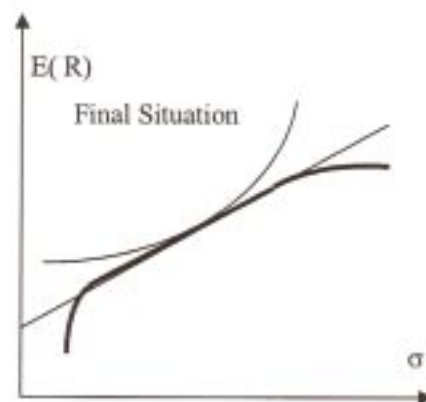
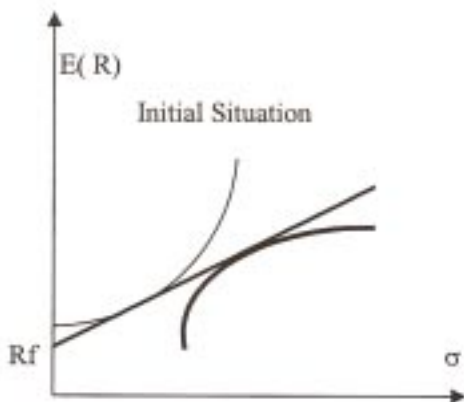
On the contrary, when a proxy is used for the market portfolio, the assumption is made that such a portfolio is efficient (it must be efficient in order to be optimal) and consequently all assets should show an exactly lineal relation between their expected returns and their Betas. In practice, it has not been possible to find this relationship: what is usually obtained is a positive but not strictly lineal relation. Some researchers have

come to think that "Beta is dead", that Beta does not explain the expected returns of different assets and that some other model must be found to get an adequate explanation.

Eugene Fama and Kenneth French (1994) found two factors that help in explaining the required rate of return: Size, and the "Book to Market" relation. Larger firms offer lower expected returns; and there are firms that are punished with a market value which is low compared with their book value. The first factor might be related with risk: the larger a firm is, its failure is less probable, because there are strong interests and enormous management capacity interested in avoiding such a thing. For the second factor, a cause has not been found: some firms are "punished" and others are "rewarded", probably because of risk factors that are not explicit.

AN ALTERNATIVE APPROACH: A STRAIGHT LINE EFFICIENT FRONTIER

In the article that originated asset valuation theory, one of the most widely quoted in financial literature and basis of the Nobel Price award, William Sharpe (1964) says that the optimal portfolio of risky assets (before the risk-free is brought into consideration) is not unique; he postulates that there is an indefinite number of different optimal portfolios, located in a section of the efficient frontier that is a straight line, implying that all those portfolios are perfectly correlated among them. This straight line segment coincides with the market line. A portfolio gets to be the "best possible one" for one person when further diversification is not useful any more, because of its perfect correlation with other efficient portfolios, which might be optimal for other individuals.



Sharpe's reasoning goes as follows: Initially, investors demand some portfolios that are optimal because, when combined with F, take them to the highest possible indifference curve. If there were homogeneous expectations, the original optimal portfolio would be the same for all investors. Any assets which are not included into those optimal portfolios are not demanded: their prices go down so that their return goes up, until some investor finds it attractive to include them into his optimal portfolio.

The process keeps going until all assets are included into at least one optimal portfolio and market for that asset is cleared. The process happens continuously in order to keep the capital market in equilibrium. The optimal portfolios thus created can be all completely different from each other, but they are all perfect substitutes of each other, because with any one of them, combined with the risk free asset, it is possible to replicate any of the other optimal portfolios.

Annex 1. Perfect correlation and portfolio effect.

If the correlation between two assets is perfect and positive, both the expected return and the standard deviation of portfolios formed with such assets are weighted averages, that is, lineal combinations of both assets. So, those portfolios will lie on the straight line that joins both assets in the $E - \sigma$ space. In the case of expected return the lineal relation is obvious:

$$P = aA + bB \quad (a + b = 1)$$

$$E(R_P) = aE(R_A) + bE(R_B)$$

In the case of the standard deviation the lineal relationship is verified as follows:

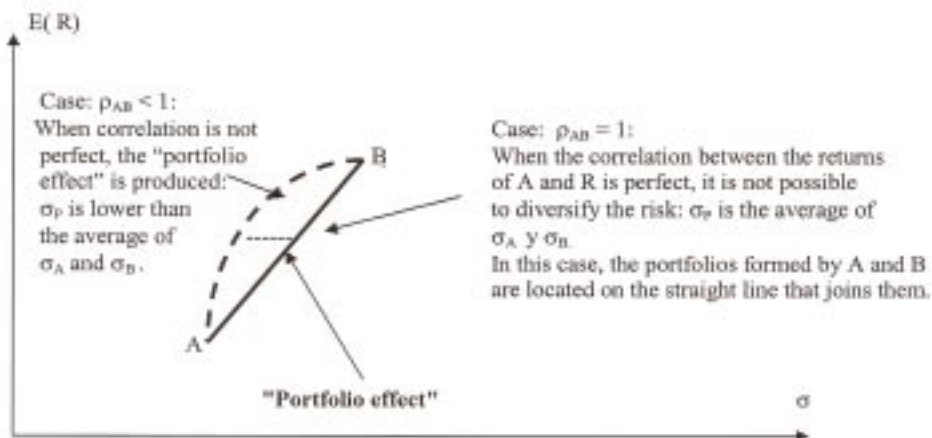
$$\sigma_P^2 = a^2 \sigma_A^2 + b^2 \sigma_B^2 + 2ab \sigma_{AB}$$

$$\sigma_P^2 = a^2 \sigma_A^2 + b^2 \sigma_B^2 + 2ab \rho_{AB} \sigma_A \sigma_B$$

When ρ_{AB} is equal to one, the former expression is the square of a binomial. When the square root is taken, it can be seen that, in this case, σ_P is the weighted average of σ_A and σ_B . In any other case σ_P is lower than the weighted average of the σ_i . And this is what is called "portfolio effect":

$$\sigma_P = a\sigma_A + b\sigma_B \quad (\text{only when } \rho_{AB} = 1)$$

$$\sigma_P < a\sigma_A + b\sigma_B \quad (\text{when } \rho_{AB} < 1)$$



RISK PREMIUM WITH A STRAIGHT LINE FRONTIER

In order to be attractive for investors, an asset must offer a rate of return at least as high as that of the risk free asset (R_F), plus a premium for risk as measured by Beta. The risk premium is the "excess return" of the optimal portfolio used to compute the Betas, that is, the yield offered by that portfolio over and above the zero risk rate (R_F). A Beta value of one means that the risk of the asset is equal to the risk of the portfolio, so that the risk premium for the asset must be the same as that of the portfolio. If the Beta is different from one, the risk premium for the asset must be proportional to its Beta:

$$E(R_A) = R_F + \beta(E(R_P) - R_F)$$

$$= R_F + \rho_{AP}\sigma_A / \sigma_P (E(R_P) - R_F) \quad (\text{See text, page 7})$$

If the frontier where the optimal portfolios are located is a straight line, as σ_P increases, the risk premium ($E(R_P) - R_F$) increases in the same proportion. For this reason, any optimal portfolio can be used to compute Betas, because the same expected return value is reached.

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